A simple model for DLA

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Abstract

A very simple model of a growing aggregate is considered. It is governed by a parameter η, and it turns out that for $\eta <= 1$ the system grows uniformly while for $\eta > 1$ the system grows almost exclusively in one direction.

Introduction

Consider the following model: we have N parallel sticks of integer length, and at integer moments exactly one stick grows by 1, with the probability of k-th stick growing proportional to a fixed power η of its length, properly normalized. Equivalently, we consider sequences of points of the first quadrant of R^n such that on each step exactly one of the coordinates of the point gets incremented by 1, and the probability of k-th coordinate being incremented is

$$(1) \qquad \frac{x_k^\eta}{\sum_{k=1}^{N} x_k^\eta}$$

The behavior of the system depends on the value of η, specifically it is different for $\eta > 1, \eta = 1, \eta < 1$.

The (slight) similarity with the standard DLA model becomes apparent if we consider the sticks as growing radially away from the center a circle.

We consider the following question: start with point (1, 1,...,1). Will the coordinates grow uniformly, or will random perturbations result in predominantly one of the coordinates growing? For example, for $\eta = \infty$ the coordinate which has grown on the first step will continue growing, while for $\eta=0$ all the coordinates will grow at equal rate. It will turn out that the coordinates will grow uniformly for $\eta \le 1$ and only one of the coordinates will grow for $\eta > 1$.

Results

Define the transition probability V(x) on the first quadrant of R^n as in formula (1):

$$(1) \qquad V_k(x) = \frac{x_k^\eta}{\sum_{k=1}^{N} x_k^\eta}, \quad k=1,2,..,N$$

Let $P_t(x,y)$ be the probability of getting from initial point x to point y in t steps. Then P satisfies the recursive equations

$$(2)\qquad P_{t+1}(x,y)=\sum_{k=1}^{N}P_t(x+e_k,y)\cdot V_k(x)$$

and

$$(3)\qquad P_{t+1}(x,y)=\sum_{k=1}^{N}P_t(x,y-e_k)\cdot V_k(y-e_k)$$

Define the first and second momenta

$$(4)\qquad M(t,x)=\sum_{y}(y-x)\cdot P_t(x,y)$$

$$(5)\qquad D(t,x)=\sum_{y}\left\|y-x\right\|^2\cdot P_t(x,y)$$

Then the following difference equations hold:

$$\begin{aligned}(6)\qquad M(t+1,x)&=\sum_{y}(y-x)\cdot P_{t+1}(x,y)=\\&=\sum_{y}(y-x)\cdot\sum_{k=1}^{N}P_t(x+e_k,y)\cdot V_k(x)=\\&=\sum_{k=1}^{N}\sum_{y}[y-(x+e_k)+e_k]\cdot V_k(x)\cdot P_t(x+e_k,y)=\\&=\sum_{k=1}^{N}\left[V_k(x)\cdot\sum_{y}[y-(x+e_k)]\cdot P_t(x+e_k,y)+V_k\cdot e_{kl}\right]=\\&=\sum_{k=1}^{N}V_k(x)\cdot M(t,x+e_k)+V(x)\end{aligned}$$

$$\begin{aligned}(7)\qquad M(t+1,x)&=\sum_{y}(y-x+e_k)\cdot\sum_{k=1}^{N}P_t(x,y)\cdot V_k(y)=\\&=\sum_{y}\left[(y-x)\cdot P_t(x,y)\cdot\sum_{k=1}^{N}V_k(y)+P_t(x,y)\cdot e_k\cdot V_k(y)\right]=\\&=M(t,x)+\sum_{y}V(y)\cdot P_t(x,y)\end{aligned}$$

$$(8)\quad D(t+1,x) = \sum_y \|y-x\|^2 \cdot P_{t+1}(x,y) =$$

$$= \sum_y \|y-x\|^2 \cdot \sum_{k=1}^{N} P_t(x+e_k,y)\cdot V_k(x) =$$

$$= \sum_{k=1}^{N} \sum_y \|y-(x+e_k)+e_k\|^2 \cdot V_k(x)\cdot P_t(x+e_k,y) =$$

$$= \sum_{k=1}^{N} \left[\begin{array}{l} V_k(x)\cdot \sum_y \|y-(x+e_k)\|^2 \cdot P_t(x+e_k,y) + \\ +2V_k(x)\cdot \left\langle \sum_y [y-(x+e_k)]\cdot P_t(x+e_k,y), e_k \right\rangle + V_k \end{array} \right] =$$

$$= \sum_{k=1}^{N} V_k(x)\cdot D(t,x+e_k) + 2\sum_{k=1}^{N} V_k(x)\cdot M_k(t,x+e_k) + 1$$

$$(9)\quad D(t+1,x) = \sum_y \sum_{k=1}^{N} \|(y+e_k)-x\|^2 \cdot P_t(x,y)\cdot V_k(y) =$$

$$= \sum_y \sum_{k=1}^{N} V_k(y)\cdot \left[\|y-x\|^2 + 2(y-x)_k + 1\right]\cdot P_t(x,y) =$$

$$= \sum_y \left[\begin{array}{l} \|y-x\|^2 \cdot P_t(x,y)\cdot \sum_{k=1}^{N} V_k(y) + 2\langle V(y),(y-x)\cdot P_t(x,y)\rangle + \\ +P_t(x,y)\cdot \sum_{k=1}^{N} V_k(y) \end{array} \right] =$$

$$= D(t,x) + 2\sum_y \langle V(y),(y-x)\rangle \cdot P_t(x,y) + 1$$

$\eta = 1$

Denote $|x|_\eta = \sum_{k=1}^{N} x_k^\eta$

For $\eta = 1$ equation (4) becomes

$$M(t+1,x) = M(t,x) + \frac{1}{|x|_1 + t}\sum_y y\cdot P_t(x,y) =$$

$$= \left(1 + \frac{1}{|x|_1 + t}\right)\cdot M(t,x) + \frac{1}{|x|_1 + t}x$$

Also, $M(0,x) = 0$

Hence $M(t,x) = \frac{t}{|x|_1}x$ is the solution

Therefore equation (6) becomes

$$D(t+1,x) = D(t,x) + 2\frac{1}{|x|_1 + t}\cdot\sum_y \langle y, y-x\rangle \cdot P_t(x,y) + 1$$

$$D(t+1,x) = D(t,x) + \frac{2}{|x|_1 + t}\cdot\sum_y \left[\|y-x\|^2 + \langle x, y-x\rangle\right]\cdot P_t(x,y) + 1$$

$$D(t+1,x) = \left(1 + \frac{2}{|x|_1 + t}\right)\cdot D(t,x) + \frac{2}{|x|_1 + t}\langle x, M(t,x)\rangle + 1$$

Substituting the formula for M, get

$$D(t+1,x) - D(t,x) = \frac{2}{|x|_1 + t}\left(D(t,x) + \frac{\|x\|^2}{|x|_1}t\right) + 1$$

If $D(t,x) = a(x)t^2 + b(x)t + c(x)$ then

$c(x) = D(0,x) = 0$

$$2a(x)t + (a(x) + b(x)) = \frac{2}{|x|_1 + t}\left(a(x)t^2 + b(x)t + \frac{\|x\|^2}{|x|_1}t\right) + 1$$

$$\left(2a(x)|x|_1 + a(x) + b(x)\right)t + (a(x) + b(x))|x|_1 = 2b(x)t + 2\frac{\|x\|^2}{|x|_1}t + t + |x|_1$$

$$\begin{cases} 2a(x)|x|_1 + a(x) - b(x) - 1 - 2\dfrac{\|x\|^2}{|x|_1} = 0 \\ a(x) + b(x) = 1 \end{cases}$$

$$\begin{cases} a(x)\cdot(2|x|_1 + 2) = 2 + 2\dfrac{\|x\|^2}{|x|_1} \\ b(x) = 1 - a(x) \end{cases}$$

$$a(x) = \frac{\|x\|^2 + |x|_1}{|x|_1\cdot\left(|x|_1 + 1\right)}$$

$$b(x) = \frac{|x|_1^2 - \|x\|^2}{|x|_1\cdot\left(|x|_1 + 1\right)}$$

$$D(t,x) = \frac{\|x\|^2 + |x|_1}{|x|_1\cdot\left(|x|_1 + 1\right)}\cdot t^2 + \frac{|x|_1^2 - \|x\|^2}{|x|_1\cdot\left(|x|_1 + 1\right)}\cdot t$$

In particular for x lying on the main diagonal

$x = (m, m, \ldots, m)$

$$M(t,x) = \left(\frac{t}{N}, \frac{t}{N}, \ldots, \frac{t}{N}\right)$$

$$D(t,x) = \frac{m+1}{Nm+1}t^2 + \frac{Nm - m}{Nm+1}t = \frac{m+1}{m + \frac{1}{N}}\cdot\left(\frac{t}{\sqrt{N}}\right)^2 + \frac{1 - \frac{1}{N}}{1 + \frac{1}{Nm}}t \approx \left(\frac{t}{\sqrt{N}}\right)^2$$

For $m = 1$

$$D(t,x) = \frac{2}{1+\frac{1}{N}} \cdot \left(\frac{t}{\sqrt{N}}\right)^2 + \frac{1-\frac{1}{N}}{1+\frac{1}{N}} t \approx 2\left(\frac{t}{\sqrt{N}}\right)^2$$

More precisely, if

$\frac{t(N)}{N} \to \infty$ as $N \to \infty$, then

$$D(t,x) \Big/ 2\left(\frac{t}{\sqrt{N}}\right)^2 \to 1 \quad \text{as} \quad N \to \infty$$

This means that on the average the coordinates of the point are uniform

Indeed,

$$D(t,x) = \sum_{y} \|y - x\|^2 \cdot P_t(x,y) = \sum_{y} \sum_{k=1}^{N} \left[\left(y_k - x_k - \frac{t}{N}\right) + \frac{t}{N}\right]^2 \cdot P_t(x,y) =$$

$$= \frac{t^2}{N} + \sum_{y} \sum_{k=1}^{N} \left[\left(y_k - x_k - \frac{t}{N}\right)\right]^2 \cdot P_t(x,y) \geq \frac{t^2}{N},$$

with equality achieved only if $y_k - x_k = \frac{t}{N}$

It follows that for $\eta < 1$ the coordinates also will be uniform

$\underline{\eta > 1}$

For $\eta > 1$, in order to use the equation (8)

$$D(t+1,x) = \sum_{k=1}^{N} V_k(x) D(t, x+e_k) + 2\sum_{k=1}^{N} V_k(x) M_k(t, x+e_k) + 1$$

we need $M_k(t,x+e_k)$

We make the following approximation:
instead of equation (6) we consider the corresponding partial differential equation:

$$M(t+1,x) = V(x) + \sum_{k=1}^{N} V_k(x) \cdot M(t, x+e_k) \Leftrightarrow$$

$$M(t+1,x) - M(t,x) = V(x) + \sum_{k=1}^{N} V_k(x) \cdot \big(M(t, x+e_k) - M(t,x)\big) \rightarrow$$

$$(9) \quad \frac{\partial M}{\partial t}(t,x) = V(x) + \sum_{k=1}^{N} V_k(x) \cdot \frac{\partial M}{\partial x_k}(t,x) \Leftrightarrow$$

$$(9) \quad \frac{\partial M}{\partial t}(t,x) - \sum_{k=1}^{N} V_k(x) \cdot \frac{\partial M}{\partial x_k}(t,x) = V(x) \Leftrightarrow$$

$\dfrac{dM}{ds}(t(s), x(s)) = V(x(s))$, where

$$\frac{dt}{ds} = 1; \quad \frac{dx_k}{ds} = -V_k(x(s))$$

Hence $\dfrac{d(M+x)}{ds}(s) = 0$

But $V_k(x) = \dfrac{x_k^\eta}{|x|_\eta}$

Hence the integral curves of the equation for x differ only in parametrisation from those of

$$\frac{dx_k}{d\tau} = -U_k(x(\tau)), \quad U_k(x) = x_k^\eta$$

$$\frac{dx_k}{d\tau} = -x_k^\eta$$

$$-x_k^{-\eta} dx_k = d\tau \quad (\eta \neq 1)$$

$$-\frac{1}{1-\eta} x_k^{1-\eta} = \tau + C_k$$

$$x_k = -(1-\eta)^{\frac{1}{1-\eta}} (\tau + C_k)^{\frac{1}{1-\eta}}$$

with $s(\tau) = \sum_{k=1}^{N} x_k(\tau)$

Thus finally

$$(M+x)_k = (1-\eta)^{\frac{1}{1-\eta}} (\tau + C_k)^{\frac{1}{1-\eta}}$$

with τ given by $t(\tau) = \sum_{k=1}^{N} M_k(\tau)$ and $C_k = C_k(x)$

Note that $\tau = \dfrac{1}{1-\eta}(M_k + x_k)^{1-\eta} - C_k$, hence

$$C_k - C_i = \frac{1}{1-\eta}(M_k + x_k)^{1-\eta} - \frac{1}{1-\eta}(M_i + x_i)^{1-\eta}$$

This is true, for example, for $t = 0$, hence

$$\frac{1}{1-\eta} x_k{}^{1-\eta} - \frac{1}{1-\eta} x_i{}^{1-\eta} = \frac{1}{1-\eta}(M_k + x_k)^{1-\eta} - \frac{1}{1-\eta}(M_i + x_i)^{1-\eta}$$

$$x_k{}^{1-\eta} - x_i{}^{1-\eta} = (M_k + x_k)^{1-\eta} - (M_i + x_i)^{1-\eta}$$

Let $\alpha = \eta - 1$

$$x_k{}^{-\alpha} - x_i{}^{-\alpha} = (M_k + x_k)^{-\alpha} - (M_i + x_j)^{-\alpha}$$

For $x = (m+1, m, m, \ldots, m)$

$m^{-\alpha} - (m+1)^{-\alpha} = (M_k + m)^{-\alpha} - (M_l + m + 1)^{-\alpha}$ and $t = M_l + (N-1)M_{kj}$

$$M_k = \frac{1}{\left(\frac{1}{m^\alpha} - \frac{1}{(m+1)^\alpha} + \frac{1}{(M_l + m + 1)^\alpha}\right)^{1/\alpha}} - m <$$

$$< \frac{1}{\left(\frac{(m+1)^\alpha - m^\alpha}{m^\alpha (m+1)^\alpha}\right)^{1/\alpha}} - m = \frac{m(m+1)}{\left((m+1)^\alpha - m^\alpha\right)^{1/\alpha}} - m$$

For $m = 1$

$$M_k < \frac{2}{\left(2^\alpha - 1\right)^{1/\alpha}} - 1$$

Denote the above constant by $C(\eta)$

Hence $M_l > t - (N-1)C(\eta)$

Since D measures non-uniformity of coordinates, for $x = (1, 1, \ldots, 1)$

$D(t, x+e_k) > D(t,x)$. Thus

$$D(t+1, x) = \sum_{k=1}^{N} V_k(x) D(t, x + e_k) + 2\sum_{k=1}^{N} V_k(x) M_k(t, x + e_k) + 1$$

$$D(t+1, x) \geq D(t, x) + 2\sum_{k=1}^{N} V_k(x) M_k(t, x + e_k) + 1$$

But it has been shown

$$M_k(t, x + e_k) > t - (N-1)C(\eta)$$

Thus $D(t+1, x) > D(t, x) + 2t - 2NC(\eta)$

Therefore

$$D(t,x) > 2\sum_{i=1}^{t-1}(i - (N-1)C(\eta)) > t^2 - t - 2t(N-1)C(\eta)$$

Thus if

$$\frac{t(N)}{N} \to \infty \quad \text{as} \quad N \to \infty,$$

$$D(t,x) \big/ t^2 \to 1 \quad \text{as} \quad N \to \infty$$

But t^2 is the maximum of D. Indeed,

$$D(t,x) = \sum_y \|y - x\|^2 \cdot P_t(x,y) \le \sum_y |y - x|_1^2 \cdot P_t(x,y) = t^2,$$

with equality achieved only if exactly one of $(y_k - x_k)$ is non - zero

Thus in this case exactly one of the coordinates is growing

Conclusions

We see that the system does indeed exhibit qualitatively different behavior for depending on whether η is greater or less than 1.