Abstract

A very simple model of a growing aggregate is considered. It is governed by a parameter $\eta$, and it turns out that for $\eta \leq 1$ the system grows uniformly while for $\eta > 1$ the system grows almost exclusively in one direction.

Introduction

Consider the following model: we have $N$ parallel sticks of integer length, and at integer moments exactly one stick grows by 1, with the probability of $k$-th stick growing proportional to a fixed power $\eta$ of its length, properly normalized. Equivalently, we consider sequences of points of the first quadrant of $\mathbb{R}^n$ such that on each step exactly one of the coordinates of the point gets incremented by 1, and the probability of $k$-th coordinate being incremented is

$$V_k(x) = \frac{x_k^\eta}{\sum_{k=1}^N x_k^\eta}$$

The behavior of the system depends on the value of $\eta$, specifically it is different for $\eta > 1, \eta = 1, \eta < 1$.

The (slight) similarity with the standard DLA model becomes apparent if we consider the sticks as growing radially away from the center a circle.

We consider the following question: start with point $(1, 1, \ldots, 1)$. Will the coordinates grow uniformly, or will random perturbations result in predominantly one of the coordinates growing? For example, for $\eta = \infty$ the coordinate which has grown on the first step will continue growing, while for $\eta = 0$ all the coordinates will grow at equal rate. It will turn out that the coordinates will grow uniformly for $\eta \leq 1$ and only one of the coordinates will grow for $\eta > 1$.

Results

Define the transition probability $V(x)$ on the first quadrant of $\mathbb{R}^n$ as in formula (1):

$$V_k(x) = \frac{x_k^\eta}{\sum_{k=1}^N x_k^\eta}, \quad k=1,2,\ldots,N$$

Let $P_t(x,y)$ be the probability of getting from initial point $x$ to point $y$ in $t$ steps. Then $P_t$ satisfies the recursive equations
(2) \[ P_{t+1}(x, y) = \sum_{k=1}^{N} P_t(x + e_k, y) \cdot V_k(x) \]

and

(3) \[ P_{t+1}(x, y) = \sum_{k=1}^{N} P_t(x, y - e_k) \cdot V_k(y - e_k) \]

Define the first and second momenta

(4) \[ M(t, x) = \sum_y P_t(x, y) \cdot (y - x) \]

(5) \[ D(t, x) = \sum_y \|y - x\|^2 \cdot P_t(x, y) \]

Then the following difference equations hold:

(6) \[ M(t + 1, x) = \sum_y (y - x) \cdot P_{t+1}(x, y) = \]

\[ = \sum_y (y - x) \cdot \sum_{k=1}^{N} P_t(x + e_k, y) \cdot V_k(x) = \]

\[ = \sum_{k=1}^{N} \sum_y |y - (x + e_k) + e_k| \cdot V_k(x) \cdot P_t(x + e_k, y) = \]

\[ = \sum_{k=1}^{N} \left[ V_k(x) \cdot \sum_y |y - (x + e_k)| \cdot P_t(x + e_k, y) + V_k \cdot e_{kl} \right] = \]

\[ = \sum_{k=1}^{N} V_k(x) \cdot M(t, x + e_k) + V(x) \]

(7) \[ M(t + 1, x) = \sum_y (y - x + e_k) \cdot \sum_{k=1}^{N} P_t(x, y) \cdot V_k(y) = \]

\[ = \sum_y \left[ (y - x) \cdot P_t(x, y) \cdot \sum_{k=1}^{N} V_k(y) + P_t(x, y) \cdot e_k \cdot V_k(y) \right] = \]

\[ = M(t, x) + \sum_y V(y) \cdot P_t(x, y) \]
\[ D(t + 1, x) = \sum_{y} \|y - x\|^2 \cdot P_{i+1}(x, y) = \]
\[ = \sum_{y} \sum_{k=1}^{N} \|y - (x + e_k) + e_k\|^2 \cdot V_k(x) \cdot P_i(x + e_k, y) = \]
\[ = \sum_{k=1}^{N} \left[ V_k(x) \cdot \sum_{y} \|y - (x + e_k)\|^2 \cdot P_i(x + e_k, y) + 2 V_k(x) \cdot \left( \sum_{y} \|y - (x + e_k)\| \cdot P_i(x + e_k, y), e_k \right) + V_k \right] = \]
\[ = \sum_{k=1}^{N} V_k(x) \cdot D(t, x + e_k) + 2 \sum_{k=1}^{N} V_k(x) \cdot M_k(t, x + e_k) + 1 \]

(9) \[ D(t + 1, x) = \sum_{y} \sum_{k=1}^{N} \|y + e_k - x\|^2 \cdot P_i(x, y) \cdot V_k(y) = \]
\[ = \sum_{y} \sum_{k=1}^{N} V_k(y) \cdot \left[ \|y - x\|^2 + 2(y - x)_k + 1 \right] \cdot P_i(x, y) = \]
\[ = \sum_{y} \left[ \|y - x\|^2 \cdot P_i(x, y) \cdot \sum_{k=1}^{N} V_k(y) + 2 \langle V(y), (y - x) \cdot P_i(x, y) \rangle + P_i(x, y) \cdot \sum_{k=1}^{N} V_k(y) \right] = \]
\[ = D(t, x) + 2 \sum_{y} \langle V(y), (y - x) \rangle \cdot P_i(x, y) + 1 \]

Denote \( |x|_\eta = \sum_{k=1}^{N} x_k^\eta \)

For \( \eta = 1 \) equation (4) becomes

\[ M(t + 1, x) = M(t, x) + \frac{1}{|x|_1 + t} \sum_{y} y \cdot P_i(x, y) = \]
\[ = \left( 1 + \frac{1}{|x|_1 + t} \right) \cdot M(t, x) + \frac{1}{|x|_1 + t} x \]

Also, \( M(0, x) = 0 \)

Hence \( M(t, x) = \frac{t}{|x|_1} x \) is the solution

Therefore equation (6) becomes
D(t+1, x) = D(t, x) + \frac{1}{|x|_1 + t} \cdot \sum_y (y - x) \cdot P_t(x, y) + 1

D(t+1, x) = D(t, x) + \frac{2}{|x|_1 + t} \cdot \sum_y \left[ \|y - x\|^2 + \langle x, y - x \rangle \right] \cdot P_t(x, y) + 1

D(t+1, x) = \left( 1 + \frac{2}{|x|_1 + t} \cdot (\|x\|^2 + \langle x, M(t, x) \rangle) \right) \cdot D(t, x) + \frac{2}{|x|_1 + t} \langle x, M(t, x) \rangle + 1

Substituting the formula for M, get

D(t+1, x) - D(t, x) = \frac{2}{|x|_1 + t} \left( D(t, x) + \frac{\|x\|^2}{|x|_1} \cdot t \right) + 1

If D(t, x) = a(x)t^2 + b(x)t + c(x) then

\begin{align*}
c(x) &= D(0, x) = 0 \\
2a(x)t + (a(x) + b(x)) &= (2a(x)\|x\|^2 + (a(x) + b(x))\|x\|) = 2b(x)t + 2\|x\|^2 + t + \|x\|_1
\end{align*}

\begin{align*}
\left\{ \\
2a(x)|x|_1 + a(x) + b(x) - b(x) - 1 - 2\|x\|^2 = 0 \\
a(x) + b(x) = 1 \\
a(x) \cdot (2|x|_1 + 2) = 2 + 2\|x\|^2 \\
b(x) = 1 - a(x)
\right. \\
a(x) = \frac{\|x\|^2 + |x|_1}{|x|_1 \cdot (|x|_1 + 1)}

b(x) = \frac{|x|_1^2 - \|x\|^2}{|x|_1 \cdot (|x|_1 + 1)}

D(t, x) = \frac{\|x\|^2 + |x|_1}{|x|_1 \cdot (|x|_1 + 1)} \cdot t^2 + \frac{|x|_1^2 - \|x\|^2}{|x|_1 \cdot (|x|_1 + 1)} \cdot t

In particular for x lying on the main diagonal

x = (m, m, ..., m)

M(t, x) = \left( \frac{t}{N}, \frac{t}{N}, ..., \frac{t}{N} \right)

D(t, x) = \frac{m + 1}{Nm + 1} \cdot t^2 + \frac{Nm - m}{Nm + 1} \cdot t = \frac{m + 1}{N} \cdot \left( \frac{t}{\sqrt{N}} \right)^2 + \frac{1 - \frac{1}{N}}{1 + \frac{1}{Nm}} \cdot t = \left( \frac{t}{\sqrt{N}} \right)^2
For $m = 1$

$$D(t, x) = \frac{2}{1 + \frac{1}{N}} \cdot \left(\frac{t}{\sqrt{N}}\right)^2 + \frac{1 - \frac{1}{N}}{1 + \frac{1}{N}} t \approx 2 \left(\frac{t}{\sqrt{N}}\right)^2$$

More precisely, if

$$\frac{t(N)}{N} \to \infty \quad \text{as} \quad N \to \infty,$$

then

$$D(t, x) \rightarrow 1 \quad \text{as} \quad N \to \infty$$

This means that on the average the coordinates of the point are uniform. Indeed,

$$D(t, x) = \sum_{y} \|y - x\|^2 \cdot P_t(x, y) = \sum_{y} \sum_{k=1}^{N} \left[\left(y_k - x_k - \frac{t}{N}\right) + \frac{t}{N}\right]^2 \cdot P_t(x, y) =$$

$$= \frac{t^2}{N} + \sum_{y} \sum_{k=1}^{N} \left[\left(y_k - x_k - \frac{t}{N}\right)\right]^2 \cdot P_t(x, y) \geq \frac{t^2}{N},$$

with equality achieved only if $y_k - x_k = \frac{t}{N}$.

It follows that for $\eta < 1$ the coordinates also will be uniform.

For $\eta > 1$, in order to use the equation (8)

$$D(t + 1, x) = \sum_{k=1}^{N} V_k(x) D(t, x + e_k) + 2 \sum_{k=1}^{N} V_k(x) M_k(t, x + e_k) + 1$$

we need $M_k(t, x + e_k)$.

We make the following approximation:

instead of equation (6) we consider the corresponding partial differential equation:
\[ M(t+1, x) = V(x) + \sum_{k=1}^{N} V_k(x) \cdot M(t, x + e_k) \]

\[ M(t+1, x) - M(t, x) = V(x) + \sum_{k=1}^{N} V_k(x) \cdot (M(t, x + e_k) - M(t, x)) \]

(9) \[ \frac{\partial M}{\partial t}(t, x) = V(x) + \sum_{k=1}^{N} V_k(x) \cdot \frac{\partial M}{\partial x_k}(t, x) \]

\[ \frac{dM}{ds}(t(s), x(s)) = V(x(s)), \text{ where } \frac{dt}{ds} = 1; \frac{dx}{ds} = -V_k(x(s)) \]

Hence \[ \frac{d(M + x)}{ds}(s) = 0 \]

But \[ V_k(x) = \frac{x_k^n}{\left|x\right|_\eta} \]

Hence the integral curves of the equation for x differ only in parameterisation from those of

\[ \frac{dx}{d\tau} = -U_k(x(\tau)), U_k(x) = x_k^n \]

\[ \frac{dx}{d\tau} = -x_k^n \]

\[ -x_k^{-\eta} dx_k = d\tau \quad (\eta \neq 1) \]

\[ \frac{1}{1-\eta} x_k^{1-\eta} = \tau + C_k \]

\[ x_k = -(1-\eta)^{1-\eta}(\tau + C_k)^{1-\eta} \]

with \( s(\tau) = \sum_{k=1}^{N} x_k(\tau) \)

Thus finally

\[ (M + x)_k = (1-\eta)^{1-\eta}(\tau + C_k)^{1-\eta} \]

with \( \tau \) given by \( t(\tau) = \sum_{k=1}^{N} M_k(\tau) \) and \( C_k = C_k(x) \)

Note that \( \tau = \frac{1}{1-\eta} (M_k + x_k)^{1-\eta} - C_k \), hence

\[ C_k - C_i = \frac{1}{1-\eta} (M_k + x_k)^{1-\eta} - \frac{1}{1-\eta} (M_i + x_i)^{1-\eta} \]
This is true, for example, for $t = 0$, hence
\[
\frac{1}{1-\eta} x_k^{1-\eta} - \frac{1}{1-\eta} x_i^{1-\eta} = \frac{1}{1-\eta} (M_k + x_k)^{1-\eta} - \frac{1}{1-\eta} (M_i + x_i)^{1-\eta}
\]
\[
x_k^{1-\eta} - x_i^{1-\eta} = (M_k + x_k)^{1-\eta} - (M_i + x_i)^{1-\eta}
\]
Let $\alpha = \eta - 1$
\[
x_k^{-\alpha} - x_i^{-\alpha} = (M_k + x_k)^{-\alpha} - (M_i + x_i)^{-\alpha}
\]
For $x = (m + 1, m, m, \ldots, m)$
\[
m^{-\alpha} - (m + 1)^{-\alpha} = (M_k + m)^{-\alpha} - (M_l + m + 1)^{-\alpha}
\]
and $t = M_l + (N - 1)M_k$
\[
M_k = \frac{1}{\left(\frac{1}{m^{\alpha}} - \frac{1}{(m + 1)^{\alpha}} + \frac{1}{(M_l + m + 1)^{\alpha}}\right)^{1/\alpha}} - m < \frac{1}{\left(\frac{(m + 1)^{\alpha} - m^{\alpha}}{m^{\alpha}(m + 1)^{\alpha}}\right)^{1/\alpha}} - m = \frac{m(m + 1)}{(m + 1)^{\alpha} - m^{\alpha})^{1/\alpha}} - m
\]
For $m = 1$
\[
M_k < \frac{2}{(2^{\alpha} - 1)^{1/\alpha}} - 1
\]
Denote the above constant by $C(\eta)$
Hence $M_l > t - (N - 1)C(\eta)$
Since $D$ measures non-uniformity of coordinates, for $x = (1, 1, \ldots, 1)$
\[
D(t, x + e_k) > D(t, x). \text{ Thus}
\]
\[
D(t + 1, x) = \sum_{k=1}^{N} V_k(x)D(t, x + e_k) + 2 \sum_{k=1}^{N} V_k(x)M_k(t, x + e_k) + 1
\]
\[
D(t + 1, x) \geq D(t, x) + 2 \sum_{k=1}^{N} V_k(x)M_k(t, x + e_k) + 1
\]
But it has been shown
\[
M_k(t, x + e_k) > t - (N - 1)C(\eta)
\]
Thus $D(t + 1, x) > D(t, x) + 2t - 2NC(\eta)$
Therefore
\[
D(t, x) > 2 \sum_{i=1}^{t-1} (i - (N - 1)C(\eta)) > t^2 - t - 2t(N - 1)C(\eta)
\]
Thus if
\[ \frac{t(N)}{N} \to \infty \quad \text{as} \quad N \to \infty, \]
\[ \frac{D(t,x)}{t^2} \to 1 \quad \text{as} \quad N \to \infty \]

But \( t^2 \) is the maximum of \( D \). Indeed,
\[ D(t,x) = \sum_{y} \|y - x\|^2 \cdot P_t(x,y) \leq \sum_{y} |y - x|_1^2 \cdot P_t(x,y) = t^2, \]

with equality achieved only if exactly one of \((y_k - x_k)\) is non-zero.

Thus in this case exactly one of the coordinates is growing.

**Conclusions**

We see that the system does indeed exhibit qualitatively different behavior for depending on whether \( \eta \) is greater or less than 1.