

Generalized chaos decomposition for Lévy processes

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February 20, 2004

$\{B(t) : t \in [0, \infty)\} = \text{Brownian motion}$, gaussian process with independent stationary increments.

On a probability space (Λ, Σ, P) with Σ generated by $\{B(t)\}$.

“Functionals” of $\{B(t)\}$: $B(1)^2 + B(2)$, $\max_{1 \leq t \leq 2} B(t)$, etc.

Square-integrable functionals: elements

$$F \in L^2(\Lambda, \Sigma, P).$$

Theorem. (Wiener?) Any such F can be written as a multiple stochastic integral

$$F = \sum_{n=0}^{\infty} \int_0^{\infty} \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, t_2, \dots, t_n) dB(t_1) dB(t_2) \dots dB(t_n),$$

where $f_n \in L^2(\mathbb{R}_+^n, dx^{\otimes n})$.

Equivalently,

$$F = \sum_{n=0}^{\infty} \int_{\mathcal{D}_n} f_n(t_1, \dots, t_n) dB(t_1) \dots dB(t_n)$$

for $f_n \in L^2(\mathcal{D}_n, dx^{\otimes n})$, where

$$\mathcal{D}_n = \{(x_1, x_2, \dots, x_n) : x_1 > x_2 > \dots > x_n > 0\}.$$

True for the (compensated) Poisson process, not true for any other Lévy process (Emery, Dermoune).

Multiple Wiener-Itô stochastic integrals: for an indicator function

$$f(t_1, \dots, t_n) = \mathbf{1}_{[a_1, b_1) \times \dots \times [a_n, b_n)}(t_1, t_2, \dots, t_n)$$

for $[a_1, b_1), \dots, [a_n, b_n)$ disjoint intervals,

$$\begin{aligned} & \int f(t_1, \dots, t_n) dB(t_1) \dots dB(t_n) \\ &= (B(b_1) - B(a_1)) \dots (B(b_n) - B(a_n)). \end{aligned}$$

Extend to simple functions in $L^2(\mathcal{D}_n, dx^{\otimes n})$ by linearity. They are dense. Extend to all of $L^2(\mathcal{D}_n, dx^{\otimes n})$ using the **Itô isometry**:

$$\begin{aligned} & \mathbb{E} \left[\left(\int f dB(t_1) \dots dB(t_n) \right) \left(\int g dB(s_1) \dots dB(s_n) \right) \right] \\ &= \langle f, g \rangle_{\mathcal{D}} \\ &= \int_{\mathcal{D}_n} f(x_1, \dots, x_n) g(x_1, \dots, x_n) dx_1 \dots dx_n. \end{aligned}$$

Here \mathbb{E} is the expectation,

$$\mathbb{E} [F] = \int_{\Lambda} F(\omega) dP(\omega).$$

A Lévy process: a process $\{X(t) : t \in [0, \infty)\}$ with stationary independent increments.

μ_t = distribution of $X(t)$. Assume $X(t)$ centered and all the moments of μ_t finite. Then

$$\log \mathcal{F}(\mu_t)(\theta) = t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x) \frac{1}{x^2} d\tau(x)$$

(Kolmogorov representation) where \mathcal{F} = Fourier transform, τ = canonical measure of the process.

Assume τ has finite moments

$$m_k(\tau) = \int_{\mathbb{R}} x^k d\tau(x) < \infty$$

and even finite exponential moments

$$\int_{\mathbb{R}} e^{\lambda x} d\tau(x) < \infty$$

for some λ .

Note: for scalar-valued measures, $\int dX dX = 0$, but for example for the Brownian motion

$$\int_0^t dX(s) dX(s) = t$$

(law of large numbers).

So let

$$\Delta_n(t) = \int_0^t (dX(s))^n.$$

be the **diagonal measures** of the process. Again assume Σ generated by $\{X(t)\}$. Then any $F \in L^2(\Lambda, \Sigma, P)$ has a representation

$$F = \sum_{n=0}^{\infty} \sum_{\vec{u}} \int f_{\vec{u}} d\Delta_{u(1)} d\Delta_{u(2)} \cdots d\Delta_{u(n)}.$$

(Nualart, Schoutens '00).

Consider the algebra \mathcal{A} generated by $\{B(t)\}$ with the inner product

$$\langle F, G \rangle = \mathbb{E} [FG] .$$

Its completion is a Hilbert space isomorphic to $L^2(\Lambda, \Sigma, P)$. Denote it by $L^2(\mathcal{A}, \mathbb{E})$.

So the Itô isometry is

$$L^2(\mathcal{D}_n, dx^{\otimes n}) \hookrightarrow L^2(\Lambda, \Sigma, P) \cong L^2(\mathcal{A}, \mathbb{E}).$$

Denote $H = L^2(\mathbb{R}_+, dx)$.

$$H^{\otimes n} = H \otimes H \otimes \dots \otimes H = L^2(\mathbb{R}_+^n, dx^{\otimes n})$$

is the “ n -particle space.”

$$\begin{aligned} \mathcal{F}_{\text{alg}}(H) &= \bigoplus_{n=0}^{\infty} H^{\otimes n} \\ &= (\mathbb{C}\Omega) \oplus H \oplus H^{\otimes 2} \oplus H^{\otimes 3} \oplus \dots \end{aligned}$$

is the algebraic Fock space. Here $\Omega =$ vacuum vector.

On $\mathcal{F}_{\text{alg}}(H)$ define the inner product: for $f \in H^{\otimes n}$, $g \in H^{\otimes k}$

$$\langle f, g \rangle = \delta_{nk} \sum_{\sigma \in \text{Sym}(n)} \int_{\mathbb{R}^n} f(t_1, \dots, t_n) g(t_{\sigma(1)}, \dots, t_{\sigma(n)}) dt_1 \dots dt_n,$$

where $\text{Sym}(n)$ is the permutation group. Quotienting out by the tensors of length 0 and completing, get the symmetric Fock space $\mathcal{F}_s(H)$.

Note that for $f, g \in L^2(\mathcal{D}_n, dx^{\otimes n})$,

$$\langle f, g \rangle = \int_{\mathcal{D}_n} f(t_1, \dots, t_n) g(t_1, \dots, t_n) dt_1 \dots dt_n,$$

so $H_s^{\otimes n} \cong L^2(\mathcal{D}_n, dx^{\otimes n})$.

For $f \in L^2(\mathbb{R}_+, dx)$, $g \in L^2(\mathcal{D}_n, dx^{\otimes n})$, define **creation** and **annihilation** operators on $\mathcal{F}_s(H)$ by

$$a^*(f)(g)(x_0, x_1, \dots, x_n) = f(x_0)g(x_1, \dots, x_n),$$

$$\begin{aligned} a(f)(g)(x_1, x_2, \dots, x_{n-1}) \\ = \sum_{k=1}^n \int_{\mathbb{R}_+} f(x_k)g(x_1, \dots, x_n) dx_k. \end{aligned}$$

$a(f)$ and $a^*(f)$ are adjoints of each other.

Let

$$\begin{aligned} a(t) &= a(\mathbf{1}_{[0,t)}), \\ a^*(t) &= a^*(\mathbf{1}_{[0,t)}), \\ X(t) &= a(t) + a^*(t). \end{aligned}$$

Define the expectation (state)

$$\varphi[A] = \langle \Omega, A\Omega \rangle.$$

$\{X(t)\}_{t \in [0, \infty)}$ is a stochastic process. In fact it is the Brownian motion, in the sense that all of its correlations with respect to the state φ are the same as for the Brownian motion.

$$\varphi [X(t_1)X(t_2) \dots X(t_n)] = \mathbb{E} [B(t_1)B(t_2) \dots B(t_n)].$$

In particular, for \mathcal{A} the algebra generated by $\{X(t)\}$,

$$L^2(\mathcal{A}, \varphi) \cong L^2(\Lambda, \Sigma, P).$$

In fact: $X(t)$'s **commute!** So the C^* -algebra generated by $\{X(t) : t \in [0, \infty)\}$ is commutative, by the Gelfand-Naimark theorem isomorphic to some $C(\Lambda)$. State φ a state on $C(\Lambda)$, so induces a measure P , which is gaussian.

Ω = cyclic and separating vector for \mathcal{A} . Therefore

$$L^2(\mathcal{A}, \varphi) \cong \mathcal{F}_s(H).$$

The isomorphism is

$$\mathcal{A} \ni A \mapsto A\Omega \in \mathcal{F}_s(H). \quad (1)$$

The Fock space has a natural grading = chaos decomposition. It remains to note that the inverse map to (1) is

$$H_s^{\otimes n} = L^2(\mathcal{D}_n, dx^{\otimes n}) \ni f$$

$$f \mapsto \int f dB(t_1)dB(t_2) \dots dB(t_n).$$

Thus the decomposition

$$\mathcal{F}_s(H) \ni F = \sum_{n=0}^{\infty} f_n$$

translates into

$$L^2(\Lambda, \Sigma, P) \ni F = \sum_{n=0}^{\infty} \int f_n dB(t_1) dB(t_2) \dots dB(t_n),$$

with

$$\|F\|_2^2 = \sum_{n=0}^{\infty} \|f_n\|_2^2$$

and $f_n \in L^2(\mathcal{D}_n, dx^{\otimes n})$.

FOCK SPACE REPRESENTATION FOR LÉVY PROCESSES.

Will do for compound Poisson processes,

$$\log \mathcal{F}(\mu_t)(\theta) = t \int_{\mathbb{R}} (e^{i\theta x} - 1) d\nu(x).$$

Roughly, $d\nu(x) = \frac{1}{x^2} d\tau(x)$. Only do this because the notation is easier.

Let $V = L^2(\mathbb{R}, \nu)$ and

$$\begin{aligned} H &= L^2(\mathbb{R}_+, dx) \otimes V \\ &= L^2(\mathbb{R}_+, dx) \otimes L^2(\mathbb{R}, \nu) = L^2(\mathbb{R}_+ \times \mathbb{R}, dx \otimes \nu). \end{aligned}$$

Define $\mathcal{F}_s(H)$. For x the independent variable in $L^2(\mathbb{R}, \nu)$, $\mathbf{1}_{[0,t)} \otimes x \in H$, so define

$$\begin{aligned} a(t) &= a(\mathbf{1}_{[0,t)} \otimes x) \\ a^*(t) &= a^*(\mathbf{1}_{[0,t)} \otimes x) \end{aligned}$$

as before.

Also, for $f \in H$ and $g \in H^{\otimes n}$, define the operator $a_0(f)$ on $\mathcal{F}_{\text{alg}}(H)$ by

$$a_0(f)(g)(x_1, \dots, x_n) = \sum_{k=1}^n f(x_k)g(x_1, \dots, x_n).$$

Define

$$a_0(t) = a_0(\mathbf{1}_{[0,t)} \otimes x).$$

Lemma. $a_0(t)$ is an essentially self-adjoint operator with a dense domain of analytic vectors.

Finally, let

$$X(t) = p_t(x) = a(t) + a^*(t) + a_0(t).$$

Proposition. The correlations of $\{X(t)\}$ are the same as those of the Lévy process with the Lévy measure ν .

Again, let \mathcal{A} be the algebra generated by $\{X(t)\}$.

Proposition. Ω is a cyclic and separating vector for \mathcal{A} , and

$$L^2(\Lambda, \Sigma, P) \cong L^2(\mathcal{A}, \varphi) \cong \mathcal{F}_s(H).$$

But this time, the stochastic integrals

$$\left(\sum_{n=0}^{\infty} \int f_n dX(t_1) dX(t_2) \dots dX(t_n) \right) \Omega$$

will only generate $\mathcal{F}_s(L^2(\mathbb{R}_+, dx) \otimes \mathbb{C})$, not the full

$$\mathcal{F}_s(L^2(\mathbb{R}_+, dx) \otimes V).$$

Fortunately,

Proposition. The diagonal measures $\int_0^t (dX(t))^k$ are

$$\Delta_k(t) = p_t(x^k) + m_k(\nu).$$

Denote by $Y_k(t)$ the centered version of $\Delta_k(t)$,

$$Y_k(t) = p_t(x^k).$$

Now the inverse map $\mathcal{F}_s(H) \rightarrow L^2(\mathcal{A}, \varphi)$, restricted to $L^2(\mathcal{D}_n, dx^{\otimes n}) \otimes V^{\otimes n}$, is given by

$$\begin{aligned} f \otimes (x_1^{u(1)} x_2^{u(2)} \dots x_n^{u(n)}) \\ \mapsto \int f dY_{u(1)}(t_1) dY_{u(2)}(t_2) \dots dY_{u(n)}(t_n). \end{aligned}$$

So every element in $L^2(\Lambda, \Sigma, P)$ can be written as this kind of stochastic integral.

To make the statement more precise, let $\{\widehat{Y}_k(t)\}$ be the Gram-Schmidt orthogonalization of $\{Y_k(t)\}$ in $L^2(\mathcal{A}, \varphi)$. Note that $\widehat{Y}_k(t) = p_t(P_k)$, where $\{P_k\}$ are the **orthogonal polynomials** with respect to ν .

For a multi-index \vec{u} , denote

$$H_{\vec{u}} = \left\{ \int f d\widehat{Y}_{u(1)}(t_1) d\widehat{Y}_{u(2)}(t_2) \dots d\widehat{Y}_{u(n)}(t_n) : \right. \\ \left. f \in L^2(\mathcal{D}_n, dx^{\otimes n}) \right\}.$$

Lemma. These subspaces are orthogonal for different \vec{u} .

We conclude that any $F \in L^2(\Lambda, \Sigma, P)$ has a unique chaos decomposition

$$F = \sum_{n=0}^{\infty} \sum_{\vec{u}} \int f_{\vec{u}} d\hat{Y}_{u(1)}(t_1) d\hat{Y}_{u(2)}(t_2) \dots d\hat{Y}_{u(n)}(t_n),$$

where

$$\|F\|_2^2 = \sum_{\vec{u}} \|f_{\vec{u}}\|_2^2$$

and $f_{\vec{u}} \in L^2(\mathcal{D}_n, dx^{\otimes n})$.

LÉVY PROCESSES ON A q -DEFORMED FULL FOCK SPACE

Let $q \in (-1, 1)$. Again start with $H = L^2(\mathbb{R}_+, dx) \otimes V$, $\xi \in V$ (“function x ”), T an operator on V (“multiplication operator by x ”). Construct $\mathcal{F}_{\text{alg}}(H)$.

But now define a new inner product

$$\begin{aligned} & \langle \xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \dots \otimes \eta_n \rangle_q \\ &= \delta_{nk} \sum_{\sigma \in \text{Sym}(n)} q^{i(\sigma)} \langle \xi_{\sigma(1)}, \eta_1 \rangle \dots \langle \xi_{\sigma(n)}, \eta_n \rangle, \end{aligned}$$

where $i(\sigma)$ the number of inversions of the permutation σ . (Bożejko, Speicher '91) \Rightarrow this is positive definite. Completing, get the q -Fock space $\mathcal{F}_q(H)$.

For $q = 1$, quotient out to the **symmetric** Fock space; for $q = -1$, quotient out to the **anti-symmetric** Fock space; for $q = 0$, get the **full** Fock space.

For $\zeta \in H$, define creation and annihilation operators on $\mathcal{F}_q(H)$ by

$$\begin{aligned} a^*(\zeta)(\eta_1 \otimes \dots \otimes \eta_n) &= \zeta \otimes \eta_1 \otimes \dots \otimes \eta_n, \\ a(\zeta)(\eta_1 \otimes \dots \otimes \eta_n) \\ &= \sum_{k=1}^n q^{k-1} \langle \zeta, \eta_k \rangle \eta_1 \otimes \dots \otimes \hat{\eta}_k \otimes \dots \otimes \eta_n. \end{aligned}$$

$a(\zeta)$ and $a^*(\zeta)$ are again adjoints of each other. Moreover,

$$a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \text{Id}.$$

Also, for S an operator on H , define the operator $a_0(S)$ on $\mathcal{F}_q(H)$ by

$$\begin{aligned} a_0(S)(\eta_1 \otimes \dots \otimes \eta_n) \\ &= \sum_{k=1}^n q^{k-1} (S\eta_k) \otimes \eta_1 \otimes \dots \otimes \hat{\eta}_k \otimes \dots \otimes \eta_n. \end{aligned}$$

$a_0(t)$ is essentially self-adjoint.

Let

$$X(t) = a(t) + a^*(t) + a_0(t)$$

Note: $X(t)$'s no longer commute, so get a “non-commutative stochastic process”. For example, without a_0 , get the q -Brownian motion.

For \mathcal{A} the algebra generated by $\{X(t)\}$, in the same way as before get:

Any $A \in L^2(\mathcal{A}, \varphi)$ has a unique chaos decomposition

$$A = \sum_{n=0}^{\infty} \sum_{\vec{u}} \int f_{\vec{u}} d\hat{Y}_{u(1)}(t_1) d\hat{Y}_{u(2)}(t_2) \dots d\hat{Y}_{u(n)}(t_n),$$

where $f_{\vec{u}} \in L^2(\mathbb{R}_+^n, dx^{\otimes n})$ and

$$\|A\|_2^2 = \sum_{\vec{u}} \|f_{\vec{u}}\|_2^2.$$

Some applications of q -Lévy processes.

- COMBINATORICS: Linearization coefficients for q -Hermite and q -Charlier polynomials, by using the q -Brownian motion and the q -Poisson process.
- PROBABILITY: Are these processes Markov? Yes for $q = 0$ (processes with freely independent increments), yes for q -Brownian motion and for the q -Poisson process. No otherwise.
- OPERATOR ALGEBRAS: What are the von Neumann algebras they generate?