

# Generalized chaos decomposition for Lévy processes

Michael Anshelevich

February 20, 2004

$\{B(t) : t \in [0, \infty)\} = \text{Brownian motion}$ , gaussian process with independent stationary increments.

On a probability space  $(\Lambda, \Sigma, P)$  with  $\Sigma$  generated by  $\{B(t)\}$ .

“Functionals” of  $\{B(t)\}$ :  $B(1)^2 + B(2)$ ,  $\max_{1 \leq t \leq 2} B(t)$ , etc.

Square-integrable functionals: elements

$$F \in L^2(\Lambda, \Sigma, P).$$

**Theorem.** (Wiener?) Any such  $F$  can be written as a multiple stochastic integral

$$F = \sum_{n=0}^{\infty} \int_0^{\infty} \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, t_2, \dots, t_n) dB(t_1) dB(t_2) \dots dB(t_n),$$

where  $f_n \in L^2(\mathbb{R}_+^n, dx^{\otimes n})$ .

Equivalently,

$$F = \sum_{n=0}^{\infty} \int_{\mathcal{D}_n} f_n(t_1, \dots, t_n) dB(t_1) \dots dB(t_n)$$

for  $f_n \in L^2(\mathcal{D}_n, dx^{\otimes n})$ , where

$$\mathcal{D}_n = \{(x_1, x_2, \dots, x_n) : x_1 > x_2 > \dots > x_n > 0\}.$$

---

True for the (compensated) Poisson process, not true for any other Lévy process (Emery, Dermoune).

Multiple Wiener-Itô stochastic integrals: for an indicator function

$$f(t_1, \dots, t_n) = \mathbf{1}_{[a_1, b_1) \times \dots \times [a_n, b_n)}(t_1, t_2, \dots, t_n)$$

for  $[a_1, b_1), \dots, [a_n, b_n)$  disjoint intervals,

$$\begin{aligned} & \int f(t_1, \dots, t_n) dB(t_1) \dots dB(t_n) \\ &= (B(b_1) - B(a_1)) \dots (B(b_n) - B(a_n)). \end{aligned}$$

Extend to simple functions in  $L^2(\mathcal{D}_n, dx^{\otimes n})$  by linearity. They are dense. Extend to all of  $L^2(\mathcal{D}_n, dx^{\otimes n})$  using the **Itô isometry**:

$$\begin{aligned} & \mathbb{E} \left[ \left( \int f dB(t_1) \dots dB(t_n) \right) \left( \int g dB(s_1) \dots dB(s_n) \right) \right] \\ &= \langle f, g \rangle_{\mathcal{D}} \\ &= \int_{\mathcal{D}_n} f(x_1, \dots, x_n) g(x_1, \dots, x_n) dx_1 \dots dx_n. \end{aligned}$$

Here  $\mathbb{E}$  is the expectation,

$$\mathbb{E} [F] = \int_{\Lambda} F(\omega) dP(\omega).$$

**A Lévy process:** a process  $\{X(t) : t \in [0, \infty)\}$  with stationary independent increments.

$\mu_t =$  distribution of  $X(t)$ . Assume  $X(t)$  centered and all the moments of  $\mu_t$  finite. Then

$$\log \mathcal{F}(\mu_t)(\theta) = t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x) \frac{1}{x^2} d\tau(x)$$

(Kolmogorov representation) where  $\mathcal{F} =$  Fourier transform,  $\tau =$  canonical measure of the process.

---

Assume  $\tau$  has finite moments

$$m_k(\tau) = \int_{\mathbb{R}} x^k d\tau(x) < \infty$$

and even finite exponential moments

$$\int_{\mathbb{R}} e^{\lambda x} d\tau(x) < \infty$$

for some  $\lambda$ .

Note: for scalar-valued measures,  $\int dX dX = 0$ , but for example for the Brownian motion

$$\int_0^t dX(s) dX(s) = t$$

(law of large numbers).

So let

$$\Delta_n(t) = \int_0^t (dX(s))^n.$$

be the **diagonal measures** of the process. Again assume  $\Sigma$  generated by  $\{X(t)\}$ . Then any  $F \in L^2(\Lambda, \Sigma, P)$  has a representation

$$F = \sum_{n=0}^{\infty} \sum_{\vec{u}} \int f_{\vec{u}} d\Delta_{u(1)} d\Delta_{u(2)} \cdots d\Delta_{u(n)}.$$

(Nualart, Schoutens '00).

Consider the algebra  $\mathcal{A}$  generated by  $\{B(t)\}$  with the inner product

$$\langle F, G \rangle = \mathbb{E} [FG] .$$

Its completion is a Hilbert space isomorphic to  $L^2(\Lambda, \Sigma, P)$ . Denote it by  $L^2(\mathcal{A}, \mathbb{E})$ .

So the Itô isometry is

$$L^2(\mathcal{D}_n, dx^{\otimes n}) \hookrightarrow L^2(\Lambda, \Sigma, P) \cong L^2(\mathcal{A}, \mathbb{E}).$$

Denote  $H = L^2(\mathbb{R}_+, dx)$ .

$$H^{\otimes n} = H \otimes H \otimes \dots \otimes H = L^2(\mathbb{R}_+^n, dx^{\otimes n})$$

is the “ $n$ -particle space.”

$$\begin{aligned} \mathcal{F}_{\text{alg}}(H) &= \bigoplus_{n=0}^{\infty} H^{\otimes n} \\ &= (\mathbb{C}\Omega) \oplus H \oplus H^{\otimes 2} \oplus H^{\otimes 3} \oplus \dots \end{aligned}$$

is the algebraic Fock space. Here  $\Omega =$  vacuum vector.

On  $\mathcal{F}_{\text{alg}}(H)$  define the inner product: for  $f \in H^{\otimes n}$ ,  $g \in H^{\otimes k}$

$$\langle f, g \rangle = \delta_{nk} \sum_{\sigma \in \text{Sym}(n)} \int_{\mathbb{R}^n} f(t_1, \dots, t_n) g(t_{\sigma(1)}, \dots, t_{\sigma(n)}) dt_1 \dots dt_n,$$

where  $\text{Sym}(n)$  is the permutation group. Quotienting out by the tensors of length 0 and completing, get the symmetric Fock space  $\mathcal{F}_s(H)$ .

Note that for  $f, g \in L^2(\mathcal{D}_n, dx^{\otimes n})$ ,

$$\langle f, g \rangle = \int_{\mathcal{D}_n} f(t_1, \dots, t_n) g(t_1, \dots, t_n) dt_1 \dots dt_n,$$

so  $H_s^{\otimes n} \cong L^2(\mathcal{D}_n, dx^{\otimes n})$ .

For  $f \in L^2(\mathbb{R}_+, dx)$ ,  $g \in L^2(\mathcal{D}_n, dx^{\otimes n})$ , define **creation** and **annihilation** operators on  $\mathcal{F}_s(H)$  by

$$a^*(f)(g)(x_0, x_1, \dots, x_n) = f(x_0)g(x_1, \dots, x_n),$$

$$\begin{aligned} a(f)(g)(x_1, x_2, \dots, x_{n-1}) \\ = \sum_{k=1}^n \int_{\mathbb{R}_+} f(x_k)g(x_1, \dots, x_n) dx_k. \end{aligned}$$

$a(f)$  and  $a^*(f)$  are adjoints of each other.

---

Let

$$\begin{aligned} a(t) &= a(\mathbf{1}_{[0,t)}), \\ a^*(t) &= a^*(\mathbf{1}_{[0,t)}), \\ X(t) &= a(t) + a^*(t). \end{aligned}$$


---

Define the expectation (state)

$$\varphi[A] = \langle \Omega, A\Omega \rangle.$$

$\{X(t)\}_{t \in [0, \infty)}$  is a stochastic process. In fact it is the Brownian motion, in the sense that all of its correlations with respect to the state  $\varphi$  are the same as for the Brownian motion.

$$\varphi [X(t_1)X(t_2) \dots X(t_n)] = \mathbb{E} [B(t_1)B(t_2) \dots B(t_n)].$$

In particular, for  $\mathcal{A}$  the algebra generated by  $\{X(t)\}$ ,

$$L^2(\mathcal{A}, \varphi) \cong L^2(\Lambda, \Sigma, P).$$

---

In fact:  $X(t)$ 's **commute!** So the  $C^*$ -algebra generated by  $\{X(t) : t \in [0, \infty)\}$  is commutative, by the Gelfand-Naimark theorem isomorphic to some  $C(\Lambda)$ . State  $\varphi$  a state on  $C(\Lambda)$ , so induces a measure  $P$ , which is gaussian.

$\Omega$  = cyclic and separating vector for  $\mathcal{A}$ . Therefore

$$L^2(\mathcal{A}, \varphi) \cong \mathcal{F}_s(H).$$

The isomorphism is

$$\mathcal{A} \ni A \mapsto A\Omega \in \mathcal{F}_s(H). \quad (1)$$

The Fock space has a natural grading = chaos decomposition. It remains to note that the inverse map to (1) is

$$H_s^{\otimes n} = L^2(\mathcal{D}_n, dx^{\otimes n}) \ni f$$

$$f \mapsto \int f dB(t_1)dB(t_2) \dots dB(t_n).$$

Thus the decomposition

$$\mathcal{F}_s(H) \ni F = \sum_{n=0}^{\infty} f_n$$

translates into

$$L^2(\Lambda, \Sigma, P) \ni F = \sum_{n=0}^{\infty} \int f_n dB(t_1) dB(t_2) \dots dB(t_n),$$

with

$$\|F\|_2^2 = \sum_{n=0}^{\infty} \|f_n\|_2^2$$

and  $f_n \in L^2(\mathcal{D}_n, dx^{\otimes n})$ .

## FOCK SPACE REPRESENTATION FOR LÉVY PROCESSES.

Will do for compound Poisson processes,

$$\log \mathcal{F}(\mu_t)(\theta) = t \int_{\mathbb{R}} (e^{i\theta x} - 1) d\nu(x).$$

Roughly,  $d\nu(x) = \frac{1}{x^2} d\tau(x)$ . Only do this because the notation is easier.

---

Let  $V = L^2(\mathbb{R}, \nu)$  and

$$\begin{aligned} H &= L^2(\mathbb{R}_+, dx) \otimes V \\ &= L^2(\mathbb{R}_+, dx) \otimes L^2(\mathbb{R}, \nu) = L^2(\mathbb{R}_+ \times \mathbb{R}, dx \otimes \nu). \end{aligned}$$

Define  $\mathcal{F}_s(H)$ . For  $x$  the independent variable in  $L^2(\mathbb{R}, \nu)$ ,  $\mathbf{1}_{[0,t)} \otimes x \in H$ , so define

$$\begin{aligned} a(t) &= a(\mathbf{1}_{[0,t)} \otimes x) \\ a^*(t) &= a^*(\mathbf{1}_{[0,t)} \otimes x) \end{aligned}$$

as before.

Also, for  $f \in H$  and  $g \in H^{\otimes n}$ , define the operator  $a_0(f)$  on  $\mathcal{F}_{\text{alg}}(H)$  by

$$a_0(f)(g)(x_1, \dots, x_n) = \sum_{k=1}^n f(x_k)g(x_1, \dots, x_n).$$

Define

$$a_0(t) = a_0(\mathbf{1}_{[0,t)} \otimes x).$$

**Lemma.**  $a_0(t)$  is an essentially self-adjoint operator with a dense domain of analytic vectors.

Finally, let

$$X(t) = p_t(x) = a(t) + a^*(t) + a_0(t).$$

**Proposition.** The correlations of  $\{X(t)\}$  are the same as those of the Lévy process with the Lévy measure  $\nu$ .

Again, let  $\mathcal{A}$  be the algebra generated by  $\{X(t)\}$ .

**Proposition.**  $\Omega$  is a cyclic and separating vector for  $\mathcal{A}$ , and

$$L^2(\Lambda, \Sigma, P) \cong L^2(\mathcal{A}, \varphi) \cong \mathcal{F}_s(H).$$

But this time, the stochastic integrals

$$\left( \sum_{n=0}^{\infty} \int f_n dX(t_1) dX(t_2) \dots dX(t_n) \right) \Omega$$

will only generate  $\mathcal{F}_s(L^2(\mathbb{R}_+, dx) \otimes \mathbb{C})$ , not the full

$$\mathcal{F}_s(L^2(\mathbb{R}_+, dx) \otimes V).$$

Fortunately,

**Proposition.** The diagonal measures  $\int_0^t (dX(t))^k$  are

$$\Delta_k(t) = p_t(x^k) + m_k(\nu).$$

Denote by  $Y_k(t)$  the centered version of  $\Delta_k(t)$ ,

$$Y_k(t) = p_t(x^k).$$

---

Now the inverse map  $\mathcal{F}_s(H) \rightarrow L^2(\mathcal{A}, \varphi)$ , restricted to  $L^2(\mathcal{D}_n, dx^{\otimes n}) \otimes V^{\otimes n}$ , is given by

$$f \otimes (x_1^{u(1)} x_2^{u(2)} \dots x_n^{u(n)}) \\ \mapsto \int f dY_{u(1)}(t_1) dY_{u(2)}(t_2) \dots dY_{u(n)}(t_n).$$

So every element in  $L^2(\Lambda, \Sigma, P)$  can be written as this kind of stochastic integral.

To make the statement more precise, let  $\{\widehat{Y}_k(t)\}$  be the Gram-Schmidt orthogonalization of  $\{Y_k(t)\}$  in  $L^2(\mathcal{A}, \varphi)$ . Note that  $\widehat{Y}_k(t) = p_t(P_k)$ , where  $\{P_k\}$  are the **orthogonal polynomials** with respect to  $\nu$ .

For a multi-index  $\vec{u}$ , denote

$$H_{\vec{u}} = \left\{ \int f d\widehat{Y}_{u(1)}(t_1) d\widehat{Y}_{u(2)}(t_2) \dots d\widehat{Y}_{u(n)}(t_n) : f \in L^2(\mathcal{D}_n, dx^{\otimes n}) \right\}.$$

**Lemma.** These subspaces are orthogonal for different  $\vec{u}$ .

We conclude that any  $F \in L^2(\Lambda, \Sigma, P)$  has a unique chaos decomposition

$$F = \sum_{n=0}^{\infty} \sum_{\vec{u}} \int f_{\vec{u}} d\hat{Y}_{u(1)}(t_1) d\hat{Y}_{u(2)}(t_2) \dots d\hat{Y}_{u(n)}(t_n),$$

where

$$\|F\|_2^2 = \sum_{\vec{u}} \|f_{\vec{u}}\|_2^2$$

and  $f_{\vec{u}} \in L^2(\mathcal{D}_n, dx^{\otimes n})$ .

## LÉVY PROCESSES ON A $q$ -DEFORMED FULL FOCK SPACE

Let  $q \in (-1, 1)$ . Again start with  $H = L^2(\mathbb{R}_+, dx) \otimes V$ ,  $\xi \in V$  (“function  $x$ ”),  $T$  an operator on  $V$  (“multiplication operator by  $x$ ”). Construct  $\mathcal{F}_{\text{alg}}(H)$ .

But now define a new inner product

$$\begin{aligned} & \langle \xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \dots \otimes \eta_n \rangle_q \\ &= \delta_{nk} \sum_{\sigma \in \text{Sym}(n)} q^{i(\sigma)} \langle \xi_{\sigma(1)}, \eta_1 \rangle \dots \langle \xi_{\sigma(n)}, \eta_n \rangle, \end{aligned}$$

where  $i(\sigma)$  the number of inversions of the permutation  $\sigma$ . (Bożejko, Speicher '91)  $\Rightarrow$  this is positive definite. Completing, get the  $q$ -Fock space  $\mathcal{F}_q(H)$ .

For  $q = 1$ , quotient out to the **symmetric** Fock space; for  $q = -1$ , quotient out to the **anti-symmetric** Fock space; for  $q = 0$ , get the **full** Fock space.

For  $\zeta \in H$ , define creation and annihilation operators on  $\mathcal{F}_q(H)$  by

$$\begin{aligned} a^*(\zeta)(\eta_1 \otimes \dots \otimes \eta_n) &= \zeta \otimes \eta_1 \otimes \dots \otimes \eta_n, \\ a(\zeta)(\eta_1 \otimes \dots \otimes \eta_n) \\ &= \sum_{k=1}^n q^{k-1} \langle \zeta, \eta_k \rangle \eta_1 \otimes \dots \otimes \hat{\eta}_k \otimes \dots \otimes \eta_n. \end{aligned}$$

$a(\zeta)$  and  $a^*(\zeta)$  are again adjoints of each other. Moreover,

$$a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \text{Id}.$$

Also, for  $S$  an operator on  $H$ , define the operator  $a_0(S)$  on  $\mathcal{F}_q(H)$  by

$$\begin{aligned} a_0(S)(\eta_1 \otimes \dots \otimes \eta_n) \\ &= \sum_{k=1}^n q^{k-1} (S\eta_k) \otimes \eta_1 \otimes \dots \otimes \hat{\eta}_k \otimes \dots \otimes \eta_n. \end{aligned}$$

$a_0(t)$  is essentially self-adjoint.

Let

$$X(t) = a(t) + a^*(t) + a_0(t)$$

Note:  $X(t)$ 's no longer commute, so get a “non-commutative stochastic process”. For example, without  $a_0$ , get the  $q$ -Brownian motion.

---

For  $\mathcal{A}$  the algebra generated by  $\{X(t)\}$ , in the same way as before get:

Any  $A \in L^2(\mathcal{A}, \varphi)$  has a unique chaos decomposition

$$A = \sum_{n=0}^{\infty} \sum_{\vec{u}} \int f_{\vec{u}} d\hat{Y}_{u(1)}(t_1) d\hat{Y}_{u(2)}(t_2) \dots d\hat{Y}_{u(n)}(t_n),$$

where  $f_{\vec{u}} \in L^2(\mathbb{R}_+^n, dx^{\otimes n})$  and

$$\|A\|_2^2 = \sum_{\vec{u}} \|f_{\vec{u}}\|_2^2.$$

Some applications of  $q$ -Lévy processes.

- COMBINATORICS: Linearization coefficients for  $q$ -Hermite and  $q$ -Charlier polynomials, by using the  $q$ -Brownian motion and the  $q$ -Poisson process.
- PROBABILITY: Are these processes Markov? Yes for  $q = 0$  (processes with freely independent increments), yes for  $q$ -Brownian motion and for the  $q$ -Poisson process. No otherwise.
- OPERATOR ALGEBRAS: What are the von Neumann algebras they generate?