Linearization coefficients for orthogonal polynomials using stochastic processes

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$\{P_n\}$ = basis for the polynomials in 1 variable.

Linearization coefficients: coefficients in the expansion of

$$P_{n_1} P_{n_2} \ldots P_{n_k}$$

in the basis $\{P_n\}$.

When $\{P_n\}$ are orthogonal,

$$P_{n_1} P_{n_2} \ldots P_{n_k} = \sum_{m=0}^{\infty} \frac{1}{\left\langle P_m^2 \right\rangle} \left\langle P_{n_1} P_{n_2} \ldots P_{n_k} P_m \right\rangle P_m,$$

where $\langle \cdot \rangle$ is the expectation with respect to the orthogonality measure,

$$\langle F \rangle = \int F(x) d\mu(x).$$

Hermite polynomials:

$$xH_n(x,t) = H_{n+1}(x,t) + ntH_{n-1}(x,t).$$

Orthogonal with respect to the Gaussian distribution

$$d\mu_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx.$$

Moments

$$\begin{aligned} m_n(\mu_t) &= \begin{cases} 0, & n \text{ odd}, \\ t^{n/2}(n-1)!!, & n \text{ even} \end{cases} \\ &= t^{n/2} \left| \{\text{Pair set partitions of } n\} \right| . \end{aligned}$$

Linearization coefficients

$$\left\langle \prod_{j=1}^{k} H_{n_k}(x,t) \right\rangle = t^{n/2} \left| \mathcal{P}_2(n_1, n_2, \ldots, n_k) \right| .$$

Here $\mathcal{P}_2(n_1, n_2, \ldots, n_k)$ = inhomogeneous pair partitions.

Centered **Charlier** polynomials:

$$xC_n(x,t) = C_{n+1}(x,t) + nC_n(x,t) + tnC_{n-1}(x,t).$$

Orthogonal with respect to the Poisson distribution

$$d\mu_t(x) = e^{-t} \sum_{j=0}^{\infty} \frac{t^j}{j!} \delta_j(x)$$

shifted by t. Moments: for $t = 1$, related to Bell numbers. More generally,

$$m_n(\mu_t) = \sum_{\substack{\pi \in \mathcal{P}(n), \\ s(\pi)=0}} t^{|\pi|}.$$

Here the sum is over all set partitions with no singletons, and $|\pi|$ = number of classes of the partition.

Linearization coefficients

$$\left\langle \prod_{j=1}^{k} C_{n_k}(x,t) \right\rangle = \sum_{\substack{\pi \in \mathcal{P}(n_1,n_2,\ldots,n_k), \\ s(\pi)=0}} t^{|\pi|}.$$

(Zeng '90)

For $\mu =$ distribution of X, its **cumulants** are coefficients in

$$\log \mathcal{F}_\mu(\theta) = \sum_{k=1}^{\infty} \frac{1}{k!} r_k (i\theta)^k.$$

Since the moments of μ are coefficients in

$$\mathcal{F}_\mu(\theta) = \sum_{k=0}^{\infty} \frac{1}{k!} m_k (i\theta)^k,$$

they are related by

$$m_n = \sum_{\pi \in \mathcal{P}(n)} R_\pi,$$

where

$$R_\pi = \prod_{B \in \pi} r_{|B|}.$$

$r_1 = m_1$, $r_2 =$ Var.

For the Gaussian distribution, $r_2 = t$, $r_k = 0$ for $k > 2$.

For the centered Poisson distribution, $r_k = t$.

Relating polynomials to processes:

$$H_n(B(t), t)$$
$$= \psi_n(B, t) = \int_{[0,t)^n} dB(t_1)dB(t_2)\dots dB(t_n),$$

where $\{B(t)\}$ is the Brownian motion, and

$$C_n(P(t), t)$$
$$= \psi_n(P, t) = \int_{[0,t)^n} dP(t_1)dP(t_2)\dots dP(t_n),$$

where $\{P(t)\}$ is the centered Poisson process.

Stochastic integrals: for simple functions

$$\mathbf{1}_{\prod_{j=1}^{n}[u(i),v(i))} \in L^2(\mathbb{R}_+^n, dx^{\otimes n})$$

with the intervals disjoint,

$$\int \mathbf{1}_{\prod_{j=1}^{n}[u(i),v(i))}(t_1, \ldots, t_n) dX_1(t_1) \ldots dX_n(t_n) \\ = \prod_{j=1}^{n} (X_j(v(j)) - X_j(u(j))).$$

The map is an isometry:

$$\mathsf{E}\left[(\int f dX_1 dX_2 \ldots dX_n)(\int f dX_1 dX_2 \ldots dX_n) \right] \\ = \langle f, g \rangle = \int f(x_1, \ldots x_n) g(x_1, \ldots x_n) dx_1 \ldots dx_n.$$

In general, approximate the integrand by simple functions, take the limit in L^2.

Note: for scalar-valued measures, $\int dX dX = 0$, but for example for the Brownian motion

$$\int_0^t dB(s) dB(s) = t$$

(law of large numbers).

Similarly, for $\{P(t)\}$ projection-valued measure,

$$\int_0^t dP(s) dP(s) = \int_0^t dP(s) = P(t).$$

So

$$\int_{\mathbb{R}_+^n} f dX(t_1) dX(t_2) \dots dX(t_n)$$

is, more precisely,

$$\int_{\mathbb{R}_+^n \setminus \text{ diagonals}} f dX(t_1) dX(t_2) \dots dX(t_n),$$

that is, approximate f by simple functions g with

$$g(t_1, t_2, \dots, t_n) = 0$$

whenever some $t_i = t_j$.

Proposition (Linearization $\Leftrightarrow$ Itô formula).

$$\prod_{j=1}^{k} \psi_{n_k} = \sum_{\pi \in \mathcal{P}(n_1, n_2, \ldots, n_k)} \mathtt{St}_\pi .$$

Here more generally, $\mathtt{St}_\pi$ = stochastic measure corresponding to the partition π (Rota, Wallstrom ’97).

$$\mathtt{St}_\pi(t) = \int_{\substack{[0,t)^l \\ \text{all } s_i\text{'s distinct}}} dX(s_{c(1)}) dX(s_{c(2)}) \cdots dX(s_{c(n)}).$$

$\Delta_n = \mathtt{St}_{\hat{1}}$ the higher diagonal measures

$$\Delta_n(t) = \int_{[0,t)} (dX(s))^n,$$

$\psi_n = \mathtt{St}_{\hat{0}}$ the full stochastic measures

$$\psi_n(t) = \int_{\substack{[0,t)^n \\ \text{all } s_i\text{'s distinct}}} dX(s_1)dX(s_2)\cdots dX(s_n).$$

Proposition . $R_\pi = \langle \mathtt{St}_\pi \rangle$ *and* $r_n = \langle \Delta_n \rangle$*.*

$$\left\langle \prod_{j=1}^{k} \psi_{n_k} \right\rangle = \sum_{\pi \in \mathcal{P}(n_1, n_2, \ldots, n_k)} R_\pi .$$

The results for Hermite and Charlier follow.

Continuous (Rogers) q-**Hermite** polynomials:

$$xH_{q,n}(x,t) = H_{q,n+1}(x,t) + t[n]_q H_{q,n-1}(x,t).$$

Here q is (say) in $(-1,1)$, and

$$[n]_q = \sum_{j=0}^{n-1} q^j = \frac{1-q^n}{1-q}.$$

Orthogonal with respect to

$$d\mu_{t,q}(x) = \frac{1}{\pi\sqrt{t}}\sqrt{1-q}\,\sin(\theta)(q;q)_\infty \left|(qe^{2i\theta};q)_\infty\right|^2 dx,$$

for $x = \frac{2}{\sqrt{1-q}}\sqrt{t}\cos(\theta)$, $\theta \in [0,\pi]$, and

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

This is a probability measure supported on the interval $[-2\sqrt{t}/\sqrt{1-q}, 2\sqrt{t}/\sqrt{1-q}]$.

Centered **continuous big q-Hermite** polynomials, which in our context are q-analogs of the Charlier polynomials. Recursion relations

$$\begin{aligned} & xC_{q,n}(x,t) \\ & = C_{q,n+1}(x,t) + [n]_q C_{q,n}(x,t) + t[n]_q C_{q,n-1}(x,t). \end{aligned}$$

$q = 1$: get Hermite and Charlier.

$q = 0$: get Chebyshev 2nd kind and a new family; for $t = 1$ orthogonal with respect to the Wishart distribution.

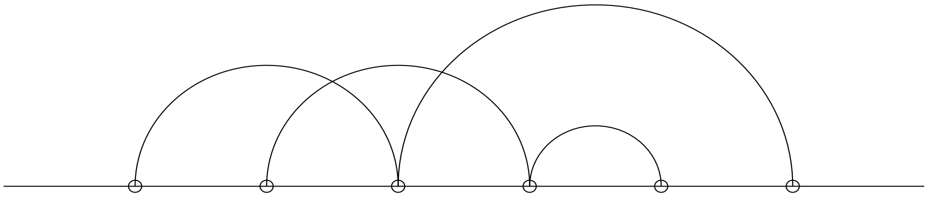
Linearization coefficients:

$$\left\langle \prod_{j=1}^{k} H_{q,n_k}(x,t) \right\rangle = \sum_{\pi \in \mathcal{P}_2(n_1,n_2,\ldots,n_k)} q^{\mathrm{c}(\pi)}$$

(Ismail, Stanton, Viennot ’87) and

$$\left\langle \prod_{j=1}^{k} C_{q,n_k}(x,t) \right\rangle = \sum_{\substack{\pi \in \mathcal{P}(n_1,n_2,\ldots,n_k) \\ s(\pi)=0}} q^{\mathrm{rc}(\pi)} t^{|\pi|}.$$

Here $\mathrm{rc}\,(\pi) =$ number of restricted crossings of the partition π.



PROCESSES ON A q-DEFORMED FULL FOCK SPACE

Consider the Hilbert space $H = L^2(\mathbb{R}_+, dx)$. Let

$$\mathcal{F}_{\mathrm{alg}}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n, dx^{\otimes n})$$

be its algebraic Fock space.

0'th component spanned by the vacuum vector Ω ("$=$" 1).

Let different components be orthogonal, and for $f, g \in L^2(\mathbb{R}_+^n, dx^{\otimes n})$,

$$\langle f, g \rangle_q = \sum_{\sigma \in \mathrm{Sym}(n)} q^{i(\sigma)} \times \int f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) g(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

(Bożejko, Speicher '91) $\Rightarrow$ this is a positive definite inner product for $q \in (-1, 1)$.

$\mathcal{F}_q(L^2(\mathbb{R}_+)) =$ the completion of $\mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+)$ with respect to the corresponding norm, the **q-deformed full Fock space**.

For $h \in L^2(\mathbb{R}_+, dx)$, define **creation**, **annihilation**, and **gauge** operators $a^*(h), a(h), p(h)$ on $\mathcal{F}_q(L^2(\mathbb{R}_+))$: for $g \in L^2(\mathbb{R}_+^n, dx^{\otimes n})$,

$$
\begin{aligned}
&a^*(h)(g)(x_1, \ldots, x_{n+1}) \\
&\quad = h(x_1) g(x_2, x_3, \ldots, x_{n+1}), \\
&a(h)(g)(x_1, \ldots, x_{n-1}) \\
&\quad = \sum_{k=1}^{n} q^{k-1} \int h(y) g(x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_{n-1}) dy, \\
&p(h)(g)(x_1, \ldots, x_n) \\
&\quad = \sum_{k=1}^{n} q^{k-1} h(x_k) g(x_k, x_1, \ldots, \widehat{x}_k, \ldots, x_n).
\end{aligned}
$$

$a(h)$ and $a^*(h)$ are adjoints of each other, $p(h) =$ self-adjoint. Moreover,

$$a(h)a^*(f) - q a^*(f) a(h) = \langle h, f \rangle \, \mathrm{Id}.$$

Let $a(t), a^*(t), p(t)$ correspond to $h = \mathbf{1}_{[0,t)}$.

The non-commutative stochastic process

$$X(t) = a^*(t) + a(t)$$

is the q**-Brownian motion**, and the process

$$X(t) = a^*(t) + a(t) + p(t)$$

is the centered q**-Poisson process**.

Corresponding distribution:

$$\langle \Omega, f(X(t))\Omega \rangle = \int f(x) d\mu_t(x).$$

For the degenerate case $q = 1$, get the corresponding classical processes. Meaning:

$$\langle \Omega, f(X(t_1), X(t_2), \ldots, X(t_n))\Omega \rangle \\ = \mathsf{E}(f(B(t_1), B(t_2), \ldots, B(t_n)))$$

and more generally, for the probability space $S =$ spectrum of $C^*(\{X(t)\})$, the map $X(t) \mapsto B(t)$ preserves norms, distributions, order, etc.

Using more general $p(t)$, can represent all Lévy processes with finite variance, or even all of them.

Proposition . *For the processes on the q-Fock space,*

$$R_\pi = q^{\mathsf{rc}(\pi)} \prod_{B \in \pi} r_{|B|}.$$

Stochastic integrals exist as limits in L^2 and give an isometry with respect to $\langle \cdot, \cdot \rangle_q$.

For the q-Brownian motion,

$$\psi_m(t) = H_{q,m}(X(t), t).$$

For the centered q-Poisson process,

$$\psi_m(t) = C_{q,m}(X(t), t).$$

Challenge for combinatorics: bijective proof for the linearization coefficients for continuous big q-Hermite.

Challenge for probability: a probabilistic proof for the Laguerre polynomials

$$xL_n(x,t) = L_{n+1}(x,t) + (2n+t)L_n(x,t) + n(t+n-1)L_{n-1}(x,t).$$

Orthogonal with respect to the Gamma distribution

$$d\mu_t(x) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} \mathbf{1}_{[0,\infty)}(x)\,dx$$

Moments:

$$m_n(\mu_t) = \sum_{\sigma \in \mathrm{Sym}(n)} t^{\mathrm{cyc}(\sigma)}.$$

Here cyc (σ) $=$ the number of cycles of σ. Linearization coefficients in terms of “generalized derangements” (Foata-Zeilberger ’88).

Have a probabilistic relation: $L_n(X(t),t)$ are martingales for the corresponding Gamma process.