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Free Appell polynomials

Michael Anshelevich

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$\mu =$ probability measure on $\mathbb{R}$, all of whose moments

$$m_n(\mu) = \int_{\mathbb{R}} x^n \, d\mu(x) < \infty.$$

Many polynomial families $\{P_n\}_{n=0}^{\infty}$ associated to it. All satisfy $P_0(x) = 1$,

$$\langle P_n, 1 \rangle = \int_{\mathbb{R}} P_n(x) \, d\mu(x) = 0$$

for $n > 0$.

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1) Orthogonal polynomials:

$$\langle P_n, P_k \rangle = \int_{\mathbb{R}} P_n(x) P_k(x) \, d\mu(x) = \delta_{n,k}.$$

Example: Hermite, Laguerre, Charlier, Jacobi, Chebyshev, etc.

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2) Appell polynomials (1880) $\{A_n\}$:

$$\partial_x A_n(x) = n A_{n-1}(x)$$

for $n > 0$. Example: $\{x^n\}$, Hermite, Bernoulli.

3) Sheffer polynomials (1937): for $U(0) = 0$, $U'(0) = 1$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) z^n = e^{xU(z) - \log M_\mu(U(z))}.$$

where

$$M_\mu(z) = \sum_{n=0}^{\infty} \frac{1}{n!} m_n(\mu) z^n$$

is the exponential moment generating function of μ. For $U(z) = z$, get Appell. Example: all Appell, Charlier, Laguerre, Abel.

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4) Kailath-Segall polynomials (1976, 2004) $\{W_n\}$: polynomials in many variables, related to the Fock space representation. Example: Hermite, Charlier.

And then...

Free analogs.

And then...

q-analogs.

Possible questions:

1) Which of the Sheffer polynomials are orthogonal?

Meixner (1934): exactly 5 families.

2) What are the linearization coefficients

$$\int_{\mathbb{R}} P_{n_1}(x) P_{n_2}(x) \dots P_{n_k}(x)\, d\mu(x)?$$

Explicit formulas for Meixner families and Appell.

Multivariate Appell polynomials

Let $\mathcal{A}$ = commutative real algebra with a unital real linear functional E. For example, $\mathcal{A} = \mathbb{R}[x_1, x_2, \ldots, x_n]$, but also $\mathcal{A}$ = random variables on some probability space, E = expectation functional.

For $X_1, X_2, \ldots, X_n \in \mathcal{A}$, define the multivariate Appell polynomial

$$\mathsf{A}_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$

by

$$\partial_i \mathsf{A}_{X_1, X_2, \ldots, X_n}(x_1, \ldots, x_i, \ldots, x_n) = \mathsf{A}_{X_1, \ldots, \widehat{X}_i, \ldots, X_n}(x_1, \ldots, \widehat{x}_i, \ldots, x_n)$$

and

$$\mathsf{E}\left[\mathsf{A}_{X_1, X_2, \ldots, X_n}(X_1, X_2, \ldots, X_n)\right] = 0.$$

Are well defined. Most of the time we will be interested in

$$\mathsf{A}_{X_1,X_2,\ldots,X_n}\left(X_1, X_2, \ldots, X_n\right),$$

which will be denoted simply by

$$\mathsf{A}\left(X_1, X_2, \ldots, X_n\right).$$

Polynomial in X's and their moments.

$$\mathsf{A}\left(X_1\right) = X_1 - \mathsf{E}\left[X_1\right],$$

$$\mathsf{A}\left(X_1, X_2\right) = X_1 X_2 - \mathsf{E}\left[X_1\right] X_2 - \mathsf{E}\left[X_2\right] X_1 \\ - \mathsf{E}\left[X_1 X_2\right] + 2\,\mathsf{E}\left[X_1\right]\mathsf{E}\left[X_2\right],$$

If instead fix $\{X_1, X_2, \ldots, X_n\}$, define the polynomials $\mathsf{A}_{\vec{u}}(\mathbf{x})$ by

$$\begin{aligned}&\mathsf{A}_{\vec{u}}(x_i : i = u(j) \text{ for some } j)\\&\quad = \mathsf{A}_{X_{u(1)}, X_{u(2)}, \ldots, X_{u(k)}}\left(x_{u(1)}, x_{u(2)}, \ldots, x_{u(k)}\right).\end{aligned}$$

Example. $\mathsf{A}_{(1,2,1)}(x_1, x_2) = \mathsf{A}_{X_1, X_2, X_1}(x_1, x_2, x_1)$.

Single-variable polynomials: for

$$\mathsf{E}[X^n] = \int_{\mathbb{R}} x^n \, d\mu(x),$$

$$A^{(n)}(x) = \mathsf{A}_{X, X, \ldots, X}(x, x, \ldots, x).$$

So

$$\partial_x A^{(n)}(x) = n A^{(n-1)}(x).$$

Free Appell polynomials

Let $\mathcal{A} =$ non-commutative real algebra with a unital real linear functional φ. For $X_1, X_2, \ldots, X_n \in \mathcal{A}$, define the monic multivariate free Appell polynomial

$$\mathsf{A}_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$$

by

$$\partial_i \mathsf{A}_{X_1,X_2,\ldots,X_n}(x_1, \ldots, x_i, \ldots, x_n)$$
$$= \mathsf{A}_{X_1,\ldots,X_{i-1}}(x_1, \ldots, x_{i-1}) \cdot \mathsf{A}_{X_{i+1},\ldots,X_n}\left(x_{i+1}, \ldots, x_n\right)$$

and

$$\varphi\left[\mathsf{A}_{X_1,X_2,\ldots,X_n}(X_1, X_2, \ldots, X_n)\right] = 0.$$

For a polynomial P in non-commuting variables

$$P(x_1, x_2, \ldots, x_n),$$

$$\partial_i x_{\vec{u}} = \sum_{j:u(j)=i} x_{u(1)} \ldots \widehat{x_{u(j)}} \ldots x_{u(k)}$$

and extending linearly.

Are well defined. Most of the time we will be interested in

$$\mathsf{A}_{X_1,X_2,\ldots,X_n}\left(X_1, X_2, \ldots, X_n\right),$$

which will be denoted simply by

$$\mathsf{A}\left(X_1, X_2, \ldots, X_n\right).$$

This is a multi-linear map.

$$\mathsf{A}\left(X_1\right) = X_1 - \varphi\left[X_1\right],$$

$$\begin{aligned}\mathsf{A}\left(X_1, X_2\right) = X_1 X_2 - \varphi\left[X_1\right] X_2 - \varphi\left[X_2\right] X_1 \\ - \varphi\left[X_1 X_2\right] + 2\varphi\left[X_1\right]\varphi\left[X_2\right],\end{aligned}$$

$$\begin{aligned}&\mathsf{A}\left(X_1, X_2, X_3\right) \\ &\quad = X_1 X_2 X_3 - \Big(\varphi\left[X_1\right] X_2 X_3 \\ &\qquad + \varphi\left[X_2\right] X_1 X_3 + \varphi\left[X_3\right] X_1 X_2\Big) \\ &\qquad - \Big(\varphi\left[X_1 X_2\right] X_3 + \varphi\left[X_2 X_3\right] X_1\Big) \\ &\qquad + \Big(2\varphi\left[X_1\right]\varphi\left[X_2\right] X_3 + \varphi\left[X_1\right]\varphi\left[X_3\right] X_2 \\ &\qquad + 2\varphi\left[X_2\right]\varphi\left[X_3\right] X_1\Big) - \varphi\left[X_1 X_2 X_3\right] \\ &\qquad + 2\varphi\left[X_1 X_2\right]\varphi\left[X_3\right] + 2\varphi\left[X_1\right]\varphi\left[X_2 X_3\right] \\ &\qquad + \varphi\left[X_1 X_3\right]\varphi\left[X_2\right] - 5\varphi\left[X_1\right]\varphi\left[X_2\right]\varphi\left[X_3\right].\end{aligned}$$

Usually, fix $\{X_1, X_2, \ldots, X_n\}$. Then define the polynomials $\mathsf{A}_{\vec{u}}(\mathbf{x})$ by

$$\mathsf{A}_{\vec{u}}\left(x_i : i = u(j) \text{ for some } j\right) \\ = \mathsf{A}_{X_{u(1)}, X_{u(2)}, \ldots, X_{u(k)}}\left(x_{u(1)}, x_{u(2)}, \ldots, x_{u(k)}\right).$$

Single-variable polynomials: $\mu =$ distribution of X,

$$A^{(n)}(x) = \mathsf{A}_{X,X,\ldots,X}(x, x, \ldots, x).$$

So

$$\partial_x A^{(n)} = \sum_{k=0}^{n-1} A^{(k)} \cdot A^{(n-k-1)}.$$

For example: Chebyshev 2nd kind.

Goal: generating function.

Joint moments of $\mathbf{X} = (X_1, X_2, \ldots, X_s) \subset \mathcal{A}$ with respect to φ are

$$M[X_{u(1)}, X_{u(2)}, \ldots, X_{u(n)}] = M[X_{\vec{u}}] = \varphi[X_{\vec{u}}]$$
$$= \varphi\left[X_{u(1)} X_{u(2)} \ldots X_{u(n)}\right],$$

and their ordinary moment-generating function is

$$\mathbf{M}(\mathbf{z}) = \sum_{\vec{u}} M[X_{\vec{u}}] z_{\vec{u}}$$
$$= \sum_i M[X_i] z_i + \sum_{i,j} M[X_i X_j] z_i z_j + \ldots.$$

Joint free cumulants of $\mathbf{X} = (X_1, X_2, \ldots, X_s)$ are defined via

$$M[X_{\vec{u}}] = \sum_{\pi \in NC(n)} \prod_{B \in \pi} R[X_{(\vec{u}:B)}],$$

Here $NC(n)$ are all the non-crossing (set) partitions of the set of n elements.

Example. For a partition $\pi = \{(1,4),(2,3)\}$, get

$$R[X_{u(1)}X_{u(4)}]R[X_{u(2)}X_{u(3)}].$$

The free cumulant generating function is

$$\mathbf{R}(\mathbf{z}) = \sum_{\vec{u}} R[X_{\vec{u}}] z_{\vec{u}}.$$

For a single random variable X with distribution μ,

$$R[X, X, \ldots, X] = r_n(\mu),$$

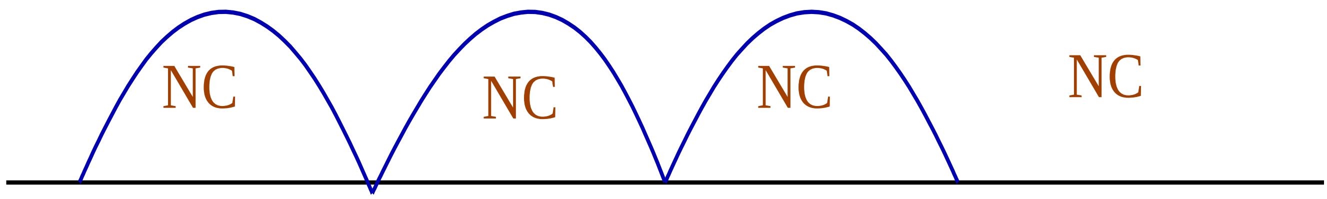
the free cumulants of μ, and $R(z) =$ R-transform.

Relation using generating functions (Nica, Speicher): let

$$z_i = w_i\Big(1 + \mathbf{M}(\mathbf{w})\Big).$$

Then

$$\mathbf{R}(\mathbf{z}) = \mathbf{R}\Big(w_1\Big(1 + \mathbf{M}(\mathbf{w})\Big), \ldots, w_n\Big(1 + \mathbf{M}(\mathbf{w})\Big)\Big)$$
$$= \mathbf{M}(\mathbf{w}).$$



Recursion relation for $A^{(n)}$ using free cumulants of μ:

$$xA^{(n)}(x) = A^{(n+1)} + \sum_{k=0}^{n} r_{n+1-k} A^{(k)}(x).$$

Theorem. Let $\{A_{\vec{u}}\}$ be the free Appell polynomials and $\mathbf{R}$ the free cumulant generating function for $\{X_1, \ldots, X_n\}$. Then

$$1 + \sum_{\vec{u}} \mathsf{A}_{\vec{u}}(\mathbf{x})\, z_{\vec{u}} = H(\mathbf{x}, \mathbf{z}) = (1 - \mathbf{x} \cdot \mathbf{z} + \mathbf{R}(\mathbf{z}))^{-1}.$$

Inverse of a power series in non-commuting variables!

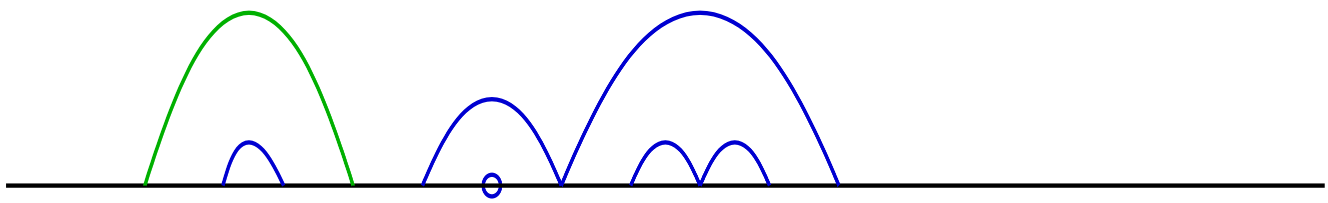
COMBINATORIAL EXPANSIONS

Fix $\vec{u} \in \mathbb{N}^n$.

Proposition. The following expansions hold.

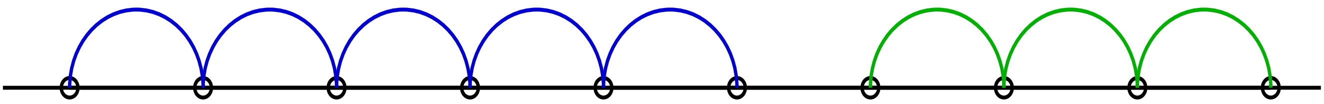
a. The expansion of $X_{\vec{u}}$ in terms of the free Appell polynomials is

$$X_{\vec{u}} = \sum_{\pi \in NC(n)} \sum_{B \in \mathrm{Outer}(\pi)} \mathsf{A}\left(X_{(\vec{u}:B)}\right) \prod_{\substack{C \in \pi, \\ C \neq B}} R[X_{(\vec{u}:C)}].$$



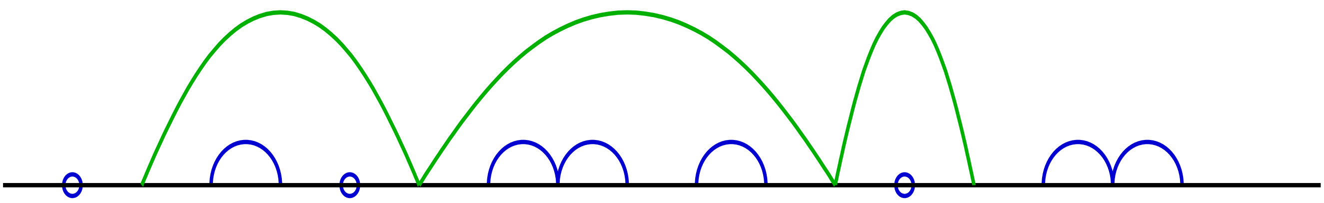
b. The recursion relation for the free Appell polynomials is

$$\mathsf{A}\left(X_{(j,\vec{u})}\right) = X_j \mathsf{A}\left(X_{\vec{u}}\right) - \sum_{k=0}^{n} R[X_j, X_{u(1)}, \ldots, X_{u(n-k)}] \times \mathsf{A}\left(X_{u(n-k+1)}, \ldots, X_{u(n)}\right).$$



c. The expansion of the free Appell polynomial is

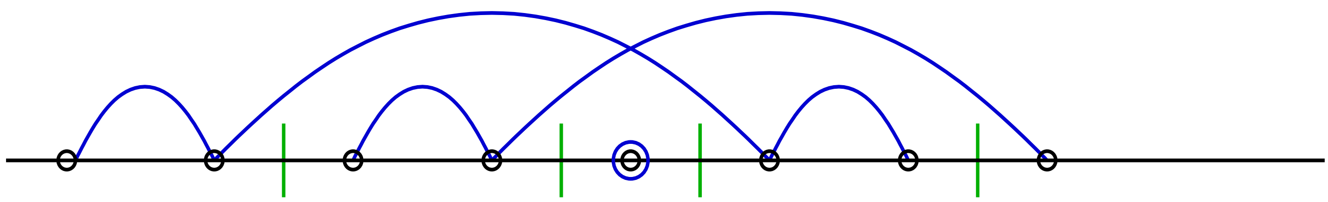
$$\mathsf{A}\left(X_{\vec{u}}\right) = \sum_{V \subset \{1,\ldots,n\}} X_{(\vec{u}:V)} \times \sum_{\substack{\pi \in Int(V^c) \\ (\pi,V) \in NC(n)}} (-1)^{|\pi|} R_{\pi}[X_{(\vec{u}:V^c)}].$$



RELATION TO FREE INDEPENDENCE:

Example. $\{X_1, X_2, X_6, X_7\}, \{X_3, X_4, X_8\}, \{X_5\}$ freely independent. Then

$$\mathsf{A}(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$
$$= \mathsf{A}(X_1, X_2) \cdot \mathsf{A}(X_3, X_4) \cdot \mathsf{A}(X_5) \cdot \mathsf{A}(X_6, X_7) \cdot \mathsf{A}(X_8)$$



Further consequence: binomial formula. For X, Y non-commuting variables,

$$(X+Y)^n = \sum X^{u(1)} Y^{u(2)} X^{u(3)} \ldots + v.v.$$

For X, Y freely independent,

$$\begin{aligned} A^{(n)}(X+Y) &= \mathsf{A}\,(X+Y, \ldots, X+Y) \\ &= \sum A^{(u(1))}(X) A^{(u(2))}(Y) A^{(u(3))}(X) \ldots + v.v. \end{aligned}$$

Theorem. Let $\vec{u}_i \in \mathbb{N}^{s(i)}$, $i = 1, 2, \ldots, k$, $N = \sum_{i=1}^{k} s(i)$, and $\vec{u} = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$. Each of the quantities

$$M[X_{\vec{u}_1}, X_{\vec{u}_2}, \ldots, X_{\vec{u}_k}] = M[X_{\vec{u}}] \tag{1}$$

$$M\Big[\mathsf{A}\left(X_{\vec{u}_1}\right), \mathsf{A}\left(X_{\vec{u}_2}\right), \ldots, \mathsf{A}\left(X_{\vec{u}_k}\right)\Big] \tag{2}$$

$$R[(X_{\vec{u}_1}), (X_{\vec{u}_2}), \ldots, (X_{\vec{u}_k})] \tag{3}$$

$$R\Big[\mathsf{A}\left(X_{\vec{u}_1}\right), \mathsf{A}\left(X_{\vec{u}_2}\right), \ldots, \mathsf{A}\left(X_{\vec{u}_k}\right)\Big] \tag{4}$$

is equal to the sum of $R_\pi[X_{\vec{u}}]$ over partitions $\pi \in NC(N)$ which are:

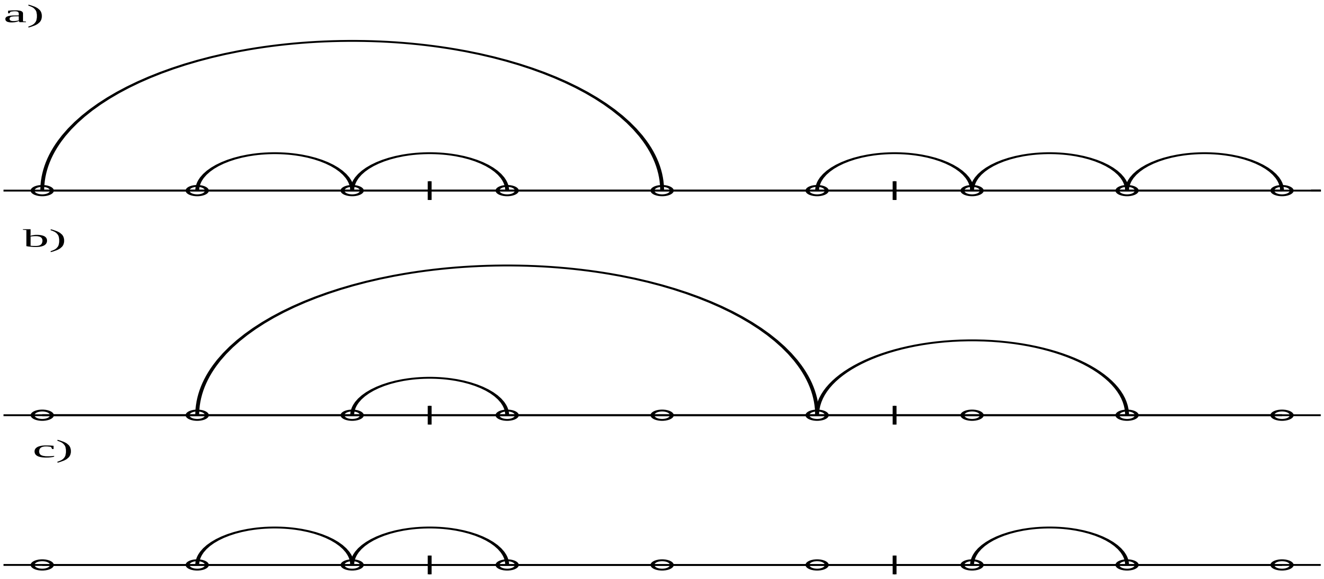
a. Equation (1): arbitrary,

b. Equation (2): non-homogeneous,

c. Equation (3): connected,

d. Equation (4): connected and non-homogeneous

with respect to the interval partition $\pi_{s(1), s(2), \ldots, s(k)}$.



With respect to the partition $\{(1,2,3),(4,5,6),(7,8,9)\}$, partitions which are (a) non-homogeneous, connected, (b) inhomogeneous, connected, (c) not connected.

Note

$$M\left[\mathsf{A}\left(X_{\vec{u}_1}\right),\mathsf{A}\left(X_{\vec{u}_2}\right),\ldots,\mathsf{A}\left(X_{\vec{u}_k}\right)\right]$$

are linearization coefficients.

Say $\{A_{\vec{u}}\}$ pseudo-orthogonal if

$$\varphi\left[A_{\vec{u}} A_{\vec{v}}\right] = 0$$

whenever $|\vec{u}| \neq |\vec{v}|$, and orthogonal if true for $\vec{u} \neq \vec{v}$.

Proposition. Fix $\{X_1, X_2, \ldots, X_n\}$ with the moment generating function $\mathbf{M}$, free cumulant generating function $\mathbf{R}$, and free Appell polynomials $\{A_{\vec{u}}\}$.

a. $\{A_{\vec{u}}\}$ are pseudo-orthogonal iff $\{X_1, X_2, \ldots, X_n\}$ form a semicircular family.

b. $\{A_{\vec{u}}\}$ are orthogonal iff $\{X_1, X_2, \ldots, X_n\}$ form a free semicircular family.

Let $\mathbf{R}$ be a free cumulant generating function, and $\mathbf{U}(\mathbf{z})$ an n-tuple of non-commutative power series such that $U_i(\mathbf{z}) = z_i +$ higher-order terms. Multivariate free Sheffer polynomials are defined via their generating function

$$H(\mathbf{x}, \mathbf{z}) = 1 + \sum_{\vec{u}} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}}$$
$$= \Big(1 - \mathbf{x} \cdot \mathbf{U}(\mathbf{z}) + \mathbf{R}(\mathbf{U}(\mathbf{z}))\Big)^{-1}.$$

When orthogonal?

In one variable, 5 families: Chebyshev polynomials and relatives (M.A. '03). In general?

Define a linear operator D_j on non-commutative power series via

$$D_j w_{u(1),\ldots,u(n)} = \begin{cases} 0, & u(n) \neq j, \\ w_{u(1),\ldots,u(n-1)}, & u(n) = j. \end{cases}$$

Define the gradient operator

$$\mathbf{D} : \mathbb{C}\langle w_1, \ldots, w_n \rangle \to \mathbb{C}\langle w_1, \ldots, w_n \rangle^n,$$

$\mathbf{D} = (D_1, D_2, \ldots, D_n)$.

Theorem. Suppose that a family of multivariate free Sheffer polynomials $\{P_{\vec{u}}\}$ is pseudo-orthogonal. Suppose also that the covariance matrix $R[X_i, X_j]$ is non-degenerate. Then

$$\mathbf{U}(\mathbf{z}) = \mathbf{F}^{<-1>}\Big(\sum_{i=1}^{n} R[X_i, X_1] z_i + \varphi[X_1], \ldots, \sum_{i=1}^{n} R[X_i, X_n] z_i + \varphi[X_n]\Big).$$

Here $F_i(\mathbf{z}) = (\mathbf{DR})(\mathbf{z}) - \varphi[X_i]$, and for an n-tuple of power series $\mathbf{F}$, $\mathbf{G} = \mathbf{F}^{<-1>}$ is the inverse of $\mathbf{F}$ under composition,

$$F_i(\mathbf{G}(\mathbf{z})) = z_i.$$

So for each $\mathbf{R}$, have a unique choice of $\mathbf{U}$. In the classical case, the corresponding condition defines the natural exponential families.