Combinatorics of Wick products

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Schrödinger representation

$$X\psi(x) = x\psi(x), \qquad X = x$$
$$P\psi(x) = -i\hbar\,\psi'(x), \qquad P = -i\hbar\,\partial_x$$

$$[X, P] = XP - PX = i\hbar\,\mathrm{Id}$$

Define the "creation and annihilation" or "raising and lowering" operators

$$A^{+} = \frac{1}{2}(X - iP) = \frac{1}{2}(x - \hbar\,\partial_x)$$
$$A^{-} = \frac{1}{2}(X + iP) = \frac{1}{2}(x + \hbar\,\partial_x)$$

Thus

$$X = A^+ + A^-, \quad P = i(A^+ - A^-).$$

Then

$$[A^-, A^+] = \frac{1}{4}[X, -iP] + \frac{1}{4}[iP, X] = (\hbar/2)\mathrm{Id}.$$

Mathematics: set $\hbar/2 = 1$.

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n-DIMENSIONAL CASE

$$X_j\psi = x_j\psi$$
$$P_j\psi = -i\partial_{x_j}\psi$$

X_j's commute among themselves.

$$A_j^+ = \frac{1}{2}(X_j - iP_j), \qquad A_j^- = \frac{1}{2}(X_j + iP_j),$$

$$[A_j^-, A_k^+] = \delta_{jk}.$$

FOCK REPRESENTATION

Let $H_0 =$ real Hilbert space, for example $\mathbb{R}^k$.

Let $H =$ its complexification, for example $\mathbb{C}^k$.

Denote $H^{\otimes n} = \underbrace{H \otimes_s H \otimes \ldots \otimes H}_{n}$, $H^{\otimes 0} = \mathbb{C}\Omega$, $\Omega =$ special vector.

$n =$ number of particles, $\Omega =$ zero particles, vacuum vector.

Define the symmetric Fock space

$$\mathcal{F}_s(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}$$
$$= \mathbb{C}\Omega \oplus H \oplus (H \otimes_s H) \oplus (H \otimes_s H \otimes_s H) \oplus \ldots.$$

A Hilbert space, with an inner product making levels orthogonal.

For $f \in H_0$, define

$$a^+(f), a^-(f) : \mathcal{F}_s(H) \to \mathcal{F}_s(H)$$

$$\begin{aligned}
&a^+(f)\,\Omega = f, \\
&a^+(f)\,(g_1 \otimes \ldots \otimes g_n) = f \otimes g_1 \otimes \ldots \otimes g_n, \\
&a^-(f)\,\Omega = 0, \\
&a^-(f)\,g = \langle f, g \rangle\,, \\
&a^-(f)\,(g_1 \otimes \ldots \otimes g_n) = \sum_{k=1}^{n} \langle f, g_k \rangle\, g_1 \otimes \ldots \otimes \widehat{g}_k \otimes \ldots \otimes g_n.
\end{aligned}$$

Note

$$\begin{aligned}
&a^-(f)a^+(h)(g_1 \otimes \ldots \otimes g_n) \\
&= \langle f, h \rangle\,(g_1 \otimes \ldots \otimes g_n) + a^+(h)a^-(f)(g_1 \otimes \ldots \otimes g_n),
\end{aligned}$$

so

$$\boxed{[a^-(f), a^+(h)] = \langle f, h \rangle\,\mathrm{Id}.}$$

If $f_1, \ldots, f_n =$ orthonormal basis for H_0,

$$[a^-(f_j), a^+(f_k)] = \left\langle f_j, f_k \right\rangle = \delta_{jk}.$$

Denote $X(f) = a^+(f) + a^-(f)$.

Thus $a^+(f), a^-(h)$ in general do not commute. But the position operators do:

$$[X(f), X(h)] = [a^+(f), a^-(h)] + [a^-(f), a^+(h)]$$
$$= \langle f, h\rangle - \langle h, f\rangle = 0$$

since f, g **real**.

Question 1.

$$\langle \Omega, X(f_1)X(f_2)\ldots X(f_n)\,\Omega\rangle = ?$$

This equals to

$$\left\langle \Omega, \left(a_+(f_1) + a_-(f_1)\right)\ldots\left(a^+(f_n) + a^-(f_n)\right)\Omega\right\rangle$$
$$= \sum_{\varepsilon\in I(n)} \left\langle \Omega, a^{\varepsilon(1)}(f_1)\ldots a^{\varepsilon(n)}(f_n)\,\Omega\right\rangle,$$

$$I(n) = \{+,-\}^n = \{\varepsilon : \varepsilon(i) = + \text{ or } -\}.$$

Question 0.

$$\left\langle \Omega, a^{\varepsilon(1)}(f_1)\ldots a^{\varepsilon(n)}(f_n)\,\Omega\right\rangle = ?$$

Example 1.

$$\left\langle \Omega, a^-(f_1)a^-(f_2)a^+(f_3)a^-(f_4)a^+(f_5)a^+(f_6)\,\Omega \right\rangle$$

$$f_6$$

$$f_5 \otimes f_6$$

$$\langle f_4, f_5\rangle\, f_6 + \langle f_4, f_6\rangle\, f_5$$

$$\langle f_4, f_5\rangle\, f_3 \otimes f_6 + \langle f_4, f_6\rangle\, f_3 \otimes f_5$$

$$\langle f_4, f_5\rangle \langle f_2, f_3\rangle\, f_6 + \langle f_4, f_6\rangle \langle f_2, f_3\rangle\, f_5 + \langle f_4, f_5\rangle \langle f_2, f_6\rangle\, f_3 + \langle f_4, f_6\rangle \langle f_2, f_5\rangle\, f_3$$

$$\langle f_4, f_5\rangle \langle f_2, f_3\rangle \langle f_1, f_6\rangle + \langle f_4, f_6\rangle \langle f_2, f_3\rangle \langle f_1, f_5\rangle + \langle f_4, f_5\rangle \langle f_2, f_6\rangle \langle f_1, f_3\rangle + \langle f_4, f_6\rangle \langle f_2, f_5\rangle \langle f_1, f_3\rangle$$

Let $\mathcal{P}_2(n)$ = a set of all pairings (perfect matchings, pair partitions, Feynman diagrams)

$$\gamma = \Big(\gamma_-(1), \gamma_+(1)\Big)\Big(\gamma_-(2), \gamma_+(2)\Big), \ldots$$

$$(4,5)(2,3)(1,6), etc.$$

Clearly $\mathcal{P}_2(\text{odd}) = \emptyset$.

A pairing $\gamma \in \mathcal{P}_2(n)$ is **consistent with** ε if

$$\varepsilon(\gamma_-(i)) = -, \qquad \varepsilon(\gamma_+(i)) = +.$$

Write $\gamma \in \mathcal{P}(\varepsilon)$.

For $\varepsilon = (-,-,+,-,+,+)$, the consistent pairings are

$$\begin{aligned} &(1,3)(2,5)(4,6), \quad (1,3)(2,6)(4,5), \\ &(1,5)(2,3)(4,6), \quad (1,6)(2,3)(4,5). \end{aligned}$$

Answer 0.

$$\begin{aligned} &\Big\langle \Omega, a^{\varepsilon(1)}(f_1) \ldots a^{\varepsilon(n)}(f_{2n})\,\Omega \Big\rangle \\ = &\sum_{\gamma \in \mathcal{P}_2(\varepsilon)} \Big\langle f_{\gamma_-(1)}, f_{\gamma_+(1)} \Big\rangle \cdots \Big\langle f_{\gamma_-(n)}, f_{\gamma_+(n)} \Big\rangle \end{aligned}$$

and 0 if the number of f's is odd.

Answer 1 (Wick formula).

$$\langle \Omega, X(f_1)X(f_2)\dots X(f_{2n})\,\Omega\rangle$$
$$= \sum_{\gamma\in\mathcal{P}_2(2n)} \left\langle f_{\gamma_-(1)}, f_{\gamma_+(1)}\right\rangle \cdots \left\langle f_{\gamma_-(n)}, f_{\gamma_+(n)}\right\rangle$$

and 0 if the number of f's is odd.

Example 2.

$$\left\langle \Omega, X(f)^{2n}\Omega \right\rangle = \|f\|^{2n}\,|\mathcal{P}_2(2n)| = \|f\|^{2n}\,\frac{(2n)!}{2^n n!}$$
$$= \int_{\mathbb{R}} x^{2n} \frac{1}{\sqrt{2\pi}\,\|f\|} e^{-x^2/2\|f\|^2}\,dx.$$

So $X(f)$ has the normal distribution with mean 0, standard deviation $\|f\|$.

WICK PRODUCTS

A Wick product

$$W(f_1, f_2, \ldots, f_n) =: X(f_1)X(f_2)\ldots X(f_n):$$

is obtained by expanding

$$X(f_1)X(f_2)\ldots X(f_n) = \sum_{\varepsilon \in I(n)} a^{\varepsilon(1)}(f_1)\ldots a^{\varepsilon(n)}(f_n)$$

and moving all a^+ to the left! Non-commutative $\Rightarrow$ get a different operator.

Example 3.

$$\begin{aligned} W(f_1, f_2) &=: X(f_1)X(f_2): \\ &=: (a_1^+ a_2^+ + a_1^+ a_2^- + a_1^- a_2^+ + a_1^- a_2^-): \\ &= a_1^+ a_2^+ + a_1^+ a_2^- + a_2^+ a_1^- + a_1^- a_2^- \\ &= X_1 X_2 - a_1^- x_2^+ + a_2^+ a_1^- = X_1 X_2 - [a_1^-, a_2^+] \\ &= X(f_1)X(f_2) - \langle f_1, f_2 \rangle \end{aligned}$$

a polynomial in $X(f_1)$, $X(f_2)$!

True in general, follows from the recursion

$$W(f, f_1, \ldots, f_n) = X(f)W(f_1, \ldots, f_n) - \sum_{k=1}^{n} \langle f, f_k \rangle \, W(f, \ldots, \widehat{f}_k, \ldots, f_n)$$

(exercise)

Also note

$$W(f_1, \ldots, f_n) \, \Omega = a^+(f_1) \ldots a^+(f_n) \, \Omega = f_1 \otimes \ldots \otimes f_n$$

since only the first term in the sum makes a non-zero contribution.

Question 2. Express $W(f_1, \ldots, f_n)$ in terms of the usual products.

Question 3. Express $X(f_1) \ldots X(f_n)$ in terms of the Wick products.

Denote by $\mathcal{P}_{2,1}(n)$ the set of all incomplete pairings (matchings, left-open pair partitions, incomplete Feynman diagrams): $\gamma \in \mathcal{P}_{2,1}(n)$ breaks $\{1, \ldots n\}$ into 2-element classes

$$(\gamma_-(i), \gamma_+(i))$$

and one-element classes $(\gamma_0(j))$. Denote by $|\gamma|$ the total number of classes in γ.

Answer 2. Using the recursion relation,

$$W(f_1, \ldots, f_n)$$
$$= \sum_{\gamma \in \mathcal{P}_{2,1}(n)} (-1)^{|\gamma|} \prod_i \left\langle f_{\gamma_-(i)}, f_{\gamma_+(i)} \right\rangle \prod_j X(f_{\gamma_0(j)}).$$

Example 4.

$$X(f_1)X(f_2)X(f_3)\,\Omega = X(f_1)X(f_2)f_3$$
$$= X(f_1)(f_2 \otimes f_3 + \langle f_2, f_3\rangle\,\Omega)$$
$$= f_1 \otimes f_2 \otimes f_3 + \langle f_1, f_2\rangle\, f_3 + \langle f_1, f_3\rangle\, f_2 + \langle f_2, f_3\rangle\, f_1.$$

In general,

$$X(f_1) \ldots X(f_n)\,\Omega$$
$$= \sum_{\gamma \in \mathcal{P}_{2,1}(n)} \prod_i \left\langle f_{\gamma_-(i)}, f_{\gamma_+(i)} \right\rangle f_{\gamma_0(1)} \otimes \ldots \otimes f_{\gamma_0(k)}.$$

Therefore

Answer 3.

$$X(f_1) \ldots X(f_n)$$
$$= \sum_{\gamma \in \mathcal{P}_{2,1}(n)} \prod_i \left\langle f_{\gamma_-(i)}, f_{\gamma_+(i)} \right\rangle W(f_{\gamma_0(1)}, \ldots, f_{\gamma_0(k)}).$$

Let $Y_p = W(f_{p,1}, \ldots, f_{p,n_p})$, $p = 1, \ldots, t$.

Answer 4.

$$\langle \Omega, Y_1 \ldots Y_t \, \Omega \rangle = \sum_{\gamma \in \mathcal{P}_2} \prod_i \left\langle f_{\gamma_-(i)}, f_{\gamma_+(i)} \right\rangle$$

where γ does not connect the elements in the same block.

Answer 5.

$$Y_1 \ldots Y_t = \sum_{\gamma \in \mathcal{P}_{2,1}} \prod_i \left\langle f_{\gamma_-(i)}, f_{\gamma_+(i)} \right\rangle W(f_{\gamma_0(1)}, \ldots, f_{\gamma_0(k)})$$

where γ does not connect the elements in the same block.

Use the normal-ordering representation for the proofs.