

Non-commutative probability I: Operator algebras background

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Let (\mathcal{A}, φ) be a non-commutative probability space.

\mathcal{A} $=^*$ -algebra with identity,

$$(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*; \quad (ab)^* = b^* a^*.$$

$\varphi =$ linear functional $\mathcal{A} \rightarrow \mathbb{C}$, a state, i.e. positive

$$\varphi [a^* a] \geq 0,$$

unital

$$\varphi [1] = 1.$$

\mathcal{A} is a C^* -algebra, has a norm $\|\cdot\|$ such that

$$\|ab\| \leq \|a\| \|b\|, \quad \|a^*\| = \|a\|,$$

and

$$\|a^* a\| = \|a\|^2.$$

$\varphi =$ norm-continuous.

Example . Let Ω be a compact set, then $\mathcal{A} = C(\Omega, \mathbb{C})$ is a C^* -algebra with complex conjugation $f^* = \bar{f}$ and supremum norm $\|f\| = \|f\|_\infty$. Note that

$$\|f\bar{f}\|_\infty = \|f\|_\infty^2.$$

$\varphi =$ expectation with respect to a probability measure P on Ω ,

$$\varphi[f] = \int_{\Omega} f(\omega) dP(\omega).$$

Theorem . Any commutative C^* -algebra is of this form.

Example . $H =$ complex Hilbert space, $\mathcal{A} =$ norm-closed $*$ -subalgebra of $\mathcal{B}(H)$, $\varphi =$ vector state

$$\varphi_\xi(a) = \langle a\xi, \xi \rangle.$$

Note

$$\|a^*a\| = \sup_{\substack{\xi \in H, \\ \|\xi\|=1}} \langle a^*a\xi, \xi \rangle = \sup_{\substack{\xi \in H, \\ \|\xi\|=1}} \langle a\xi, a\xi \rangle = \|a\|^2.$$

Theorem . Any C^* -algebra is of this form.

Assume the state φ to be faithful: $\varphi [a^*a] = 0$ only if $a = 0$ (“full support”).

GELFAND-NAIMARK-SEGAL (GNS) CONSTRUCTION.

Imitate representing $C(\Omega)$ on $L^2(\Omega, P)$.

Put on \mathcal{A} the inner product $\langle a, b \rangle = \varphi [b^*a]$. Faithful state \Rightarrow get a norm. Denote by $L^2(\mathcal{A}, \varphi)$ the completion of \mathcal{A} with respect to this norm, by \tilde{a} the element of $L^2(\mathcal{A}, \varphi)$ corresponding to a . Represent \mathcal{A} on $L^2(\mathcal{A}, \varphi)$ by

$$L_a \tilde{b} = (ab)\tilde{\gamma}.$$

Extend to all of $L^2(\mathcal{A}, \varphi)$ with $\|L_a\| = \|a\|$.

Let $\xi = \tilde{1}$. $\langle L_a \xi, \xi \rangle = \langle a, 1 \rangle = \varphi [a]$.

Thus: “any abstract C^* -algebra is a concrete C^* -algebra”.

To do probability, want an analog of the $L^\infty(\Omega)$.

A von Neumann algebra = *-subalgebra of $\mathcal{B}(H)$ closed in the topology of pointwise convergence (strong topology),

$$T_\alpha \in \mathcal{A}, \forall \xi, T_\alpha \xi \rightarrow T_\xi \text{ then } T \in \mathcal{A}.$$

φ normal (continuous with respect to the σ -weak topology).

Every von Neumann algebra is a C^* -algebra. But intuitively quite different.

Example . Let $(\Omega, \Sigma, P) = \sigma$ -finite measure space, then $L^\infty(\Omega, P) \subset \mathcal{B}(L^2(\Omega, P))$ is a von Neumann algebra.

Theorem . Every commutative von Neumann algebra is such.

Other examples: $M_n(\mathbb{C}), \mathcal{B}(H)$.

Theorem (Double commutant theorem). Denote by \mathcal{A}' the commutant of \mathcal{A} ,

$$\mathcal{A}' = \{b \in \mathcal{B}(H) : ab = ba \text{ for all } a \in \mathcal{A}\}.$$

If \mathcal{A} is a von Neumann algebra, $\mathcal{A}'' = \mathcal{A}$.

Note obviously $\mathcal{A} \subset \mathcal{A}''$.

Example (Group algebras). $G =$ discrete group, e.g. finite, \mathbb{Z} , \mathbb{Z}^n , Sym_∞ , \mathbb{F}_n , $SL(n, \mathbb{Z})$.

$l^2(G) =$ square-summable functions on G , $\sum_{x \in G} |f(x)|^2 < \infty$.

Represent G on $l^2(G)$ by

$$(L_x f)(y) = f(x^{-1}y).$$

More generally, represent $l^1(G)$ on $l^2(G)$ by

$$(L_g f)(y) = \sum_{x \in G} g(x) f(x^{-1}y) = \sum_{xz=y} g(x) f(z)$$

(convolution). Note $L_x = L_{\delta_x}$.

$C_r^*(G) =$ norm-closure of $L_{l^1(G)}$, the (reduced) group C^* -algebra.

$W^*(G) =$ strong closure of $L_{l^1(G)}$, the group von Neumann algebra.

State: evaluation at the identity

$$\varphi [g] = g(e) = \langle L_g \delta_e, \delta_e \rangle$$

which extends to the von Neumann algebra.

Probability space (Ω, Σ, P) , subsets = events, $P(U)$ = probability of U .

Instead, talk about random variables = measurable *real-valued* functions, $X \in L^\infty(\Omega, P)$ and the expectation functional $E_P(X) = \int_\Omega X(\omega) dP(\omega)$.

Events $U \leftrightarrow \mathbf{1}_U$, $P(U) = E_P(\mathbf{1}_U)$.

Quantum events = subspaces of $H \leftrightarrow$ projections in $\mathcal{B}(H)$,

$$p = p^2 = p^*.$$

$\varphi [p] \leftrightarrow$ probability.

Quantum random variables = observables = hermitian operators.

Look at (\mathcal{A}, φ) , projections in \mathcal{A} .

Assume \mathcal{A} is a factor, has a trivial center.

Subspaces of the same size related by isomorphism. Same for projections (via partial isometries).

(Partial order) modulo (equivalence relation \sim) gives complete order. These ordered sets can be

$\{0, 1, \dots, n\}$	$M_n(\mathbb{C})$
$\{0, 1, \dots, \infty\}$	$\mathcal{B}(H)$
$[0, 1]$	II_1 -factors
$[0, \infty]$	II_∞ -factors
$\{0, \infty\}$	III -factors

Here an infinite projection is equivalent to a sub-projection.

II_1 -factors have “continuous dimensions”. If \mathcal{A} is a II_1 -factor, it has a unique trace functional φ ,

$$\varphi [ab] = \varphi [ba]$$

with values in $[0, 1]$. For $\mathcal{A} = M_n(\mathbb{C})$, $\text{tr} = \frac{1}{n} \text{Tr}$ takes values in $\left\{0, \frac{1}{n}, \dots, \frac{n}{n} = 1\right\}$. So they are better generalizations of M_n than $\mathcal{B}(H)$.

Random variables, real values \leftrightarrow hermitian (self-adjoint).

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(E.g. Gaussian)

$$(\Omega, P) \xrightarrow{X} \mathbb{R},$$

$$\mu_X = X_*P,$$

i.e. $\mu_X(U) = P(X \in U)$.

Another way: $E_P(X^n) = \int_{\mathbb{R}} x^n d\mu_X(x)$.

(\mathcal{A}, φ) qp space, $\varphi(X^n) = \int_{\mathbb{R}} x^n d\mu_X(x)$. More general way: via the Spectral Theorem.

$$X = \int_{\mathbb{R}} x dE(x),$$

E = projection-valued measure.

$$f(X) = \int_{\mathbb{R}} f(x) dE(x),$$

$$\varphi(X^n) = \int x^n d\varphi [E(x)],$$

so $\mu_X(U) = \varphi [E(U)]$.

$H = \text{real Hilbert space, finite or } \infty\text{-dim.}$

$H_{\mathbb{C}} = \text{its complexification.}$

Example: $\mathbb{R}^n \subset \mathbb{C}^n, L^2(\mathbb{R}_+, \mathbb{R}) \subset L^2(\mathbb{R}_+, \mathbb{C}).$

$$H_{\mathbb{C}}^{\otimes n} = \underbrace{H_{\mathbb{C}} \otimes H_{\mathbb{C}} \otimes \dots \otimes H_{\mathbb{C}}}$$

is the “ n -particle space.”

$$\begin{aligned} \mathcal{F}_{\text{alg}}(H_{\mathbb{C}}) &= \bigoplus_{i=0}^{\infty} H_{\mathbb{C}}^{\otimes i} \\ &= (\mathbb{C}\Omega) \oplus H_{\mathbb{C}} \oplus H_{\mathbb{C}}^{\otimes 2} \oplus H_{\mathbb{C}}^{\otimes 3} \oplus \dots \end{aligned}$$

is the algebraic Fock space. Here $\Omega = \text{vacuum vector.}$

On $\mathcal{F}_{\text{alg}}(H_{\mathbb{C}})$ define the inner product

$$\begin{aligned} \langle \xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \dots \otimes \eta_k \rangle_0 \\ = \delta_{nk} \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle \dots \langle \xi_n, \eta_n \rangle. \end{aligned}$$

Define the operator P_1 by

$$P_1(\xi_1 \otimes \dots \otimes \xi_n) = \sum_{\sigma \in \text{Sym}(n)} \xi_{\sigma(1)} \otimes \dots \otimes \xi_{\sigma(n)},$$

where $\text{Sym}(n)$ is the permutation group.

P_1 on the n 'th component $= n! \times$ projection onto the symmetric tensors.

Define the inner product

$$\langle \bar{\xi}, \bar{\eta} \rangle_1 = \langle \bar{\xi}, P_1 \bar{\eta} \rangle_0$$

and the symmetric Fock space $\mathcal{F}_1(H_{\mathbb{C}})$ as the completion of

$$\mathcal{F}_{\text{alg}}(H_{\mathbb{C}}) / \text{kernel}(P_1)$$

with respect to the corresponding norm.

For $\xi \in H$, define creation and annihilation operators on $\mathcal{F}_1(H_{\mathbb{C}})$ by

$$\begin{aligned}
 a^*(\xi)(\Omega) &= \xi, \\
 a^*(\xi)(\eta_1 \otimes \dots \otimes \eta_n) &= \xi \otimes \eta_1 \otimes \dots \otimes \eta_n, \\
 a(\xi)(\Omega) &= 0, \\
 a(\xi)(\eta) &= \langle \xi, \eta \rangle \Omega, \\
 a(\xi)(\eta_1 \otimes \dots \otimes \eta_n) \\
 &= \sum_{k=1}^n \langle \xi, \eta_k \rangle \eta_1 \otimes \dots \otimes \hat{\eta}_k \otimes \dots \otimes \eta_n.
 \end{aligned}$$

$a(\xi)$ and $a^*(\xi)$ are adjoints of each other.

Commutation relations:

$$[a(\xi), a^*(\eta)] = a(\xi)a^*(\eta) - a^*(\eta)a(\xi) = \langle \xi, \eta \rangle \text{Id}$$

and a 's commute with a 's, a^* 's with a^* 's.

Let $X(\xi) = a(\xi) + a^*(\xi)$, $P(\xi) = i(a^*(\xi) - a(\xi))$,

$$[X(\xi), P(\eta)] = 2i \langle \xi, \eta \rangle \text{Id}.$$

and X 's commute with X 's, P 's commute with P 's,

Now let $H = L^2(\mathbb{R}_+, dx)$ and

$$\begin{aligned} a(t) &= a(\mathbf{1}_{[0,t]}), \\ a^*(t) &= a^*(\mathbf{1}_{[0,t]}), \\ X(t) &= a(t) + a^*(t). \end{aligned}$$

Then $\{X(t)\}_{t \in [0, \infty)}$ is a stochastic process, moreover a commutative one.

Let φ be the vacuum state $\varphi [T] = \langle \Omega, T\Omega \rangle$.

$$\varphi [X(t)^n] = \sum \langle a^{\varepsilon(1)} a^{\varepsilon(2)} \dots a^{\varepsilon(n)} \Omega, \Omega \rangle,$$

where each $\varepsilon(i)$ is either $*$ or nothing.

$$\begin{aligned} \varphi [X(t)^n] &= t^{n/2} \# \{\text{pair partitions of } n\} \\ &= \begin{cases} 0, & n \text{ odd,} \\ t^{n/2} (n-1)!!, & n \text{ even.} \end{cases} \\ &= \int_{\mathbb{R}} x^n d\mu_t(x). \end{aligned}$$

Here

$$\mu_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

is the Gaussian density.

Moreover, the increments of the process are independent, that is for $t_1 < t_2 < \dots < t_n$,

$$\begin{aligned} & \varphi [(X(t_2) - X(t_1)) \dots (X(t_n) - X(t_{n-1}))] \\ &= \varphi [(X(t_2) - X(t_1))] \dots \varphi [(X(t_n) - X(t_{n-1}))] \end{aligned}$$

and these increments commute.

Therefore $\{X(t)\} =$ Brownian motion.

The algebra generated by $\{X(t)\}$ is commutative.

Define the operator P_{-1} by

$$P_{-1}(\xi_1 \otimes \dots \otimes \xi_n) = \sum_{\sigma \in \text{Sym}(n)} (-1)^{i(\sigma)} \xi_{\sigma(1)} \otimes \dots \otimes \xi_{\sigma(n)},$$

where $i(\sigma)$ is the number of inversions of σ ,
and $(-1)^{i(\sigma)}$ is the parity of σ .

P_{-1} on the n 'th component = $n! \times$ projection onto the anti-symmetric tensors.

Define the inner product $\langle \bar{\xi}, \bar{\eta} \rangle_{-1} = \langle \bar{\xi}, P_{-1} \bar{\eta} \rangle_0$ and the anti-symmetric Fock space $\mathcal{F}_{-1}(H_{\mathbb{C}})$ as the completion of

$$\mathcal{F}_{\text{alg}}(H_{\mathbb{C}}) / \text{kernel}(P_{-1})$$

with respect to the corresponding norm.

For $\xi \in H$, define creation operators as before, and annihilation operators by

$$\begin{aligned} a(\xi)(\eta_1 \otimes \dots \otimes \eta_n) \\ = \sum_{k=1}^n (-1)^{k-1} \langle \xi, \eta_k \rangle \eta_1 \otimes \dots \otimes \hat{\eta}_k \otimes \dots \otimes \eta_n. \end{aligned}$$

Again $a(\xi)$ and $a^*(\xi)$ are adjoints of each other.

Commutation relations:

$$[a(\xi), a^*(\eta)]_+ = a(\xi)a^*(\eta) + a^*(\eta)a(\xi) = \langle \xi, \eta \rangle \text{Id}$$

and a 's anti-commute with a 's, a^* 's with a^* 's. Also, for $X(\xi) = a(\xi) + a^*(\xi)$, $P(\xi) = i(a^*(\xi) - a(\xi))$,

$$[X(\xi), P(\eta)]_+ = 2i \langle \xi, \eta \rangle \text{Id}$$

and X 's anti-commute with X 's, P 's anti-commute with P 's,

The algebra generated by $\{X(t)\}$ is non-commutative. For H finite-dimensional, it is a Clifford algebra. For H infinite-dimensional, the von Neumann algebra is the hyperfinite II_1 -factor.

Now consider the inner product $\langle \cdot, \cdot \rangle_0$, that is, $P_0 = \text{identity}$. The completion of $\mathcal{F}_{\text{alg}}(H_{\mathbb{C}})$ is the full Fock space $\mathcal{F}_0(H_{\mathbb{C}})$.

For $\xi \in H$, define creation operators as before, and annihilation operators by

$$a(\xi)(\eta_1 \otimes \dots \otimes \eta_m) = \langle \xi, \eta_1 \rangle \eta_2 \otimes \dots \otimes \eta_m.$$

Again, $a(\xi)$ and $a^*(\xi)$ are adjoints of each other.

Commutation relations:

$$a(\xi)a^*(\eta) = \langle \xi, \eta \rangle \text{Id}$$

and that's it.

For $H = L^2(\mathbb{R}_+, dx)$, define $\{X(t)\}_{t \in [0, \infty)}$. Another non-commutative stochastic process.

$$\varphi [X(t)^n] = \sum \langle a^{\varepsilon(1)} a^{\varepsilon(2)} \dots a^{\varepsilon(n)} \Omega, \Omega \rangle,$$

where each $\varepsilon(i)$ is either $*$ or nothing.

$$\begin{aligned} \varphi [X(t)^n] &= t^{n/2} \# \{ \text{non-crossing pair partitions of } n \} \\ &= \begin{cases} 0, & n \text{ odd,} \\ t^{n/2} \text{ Catalan number } c_n, & n \text{ even.} \end{cases} \\ &= \int_{\mathbb{R}} x^n d\nu_t(x), \end{aligned}$$

where

$$\nu_t(x) = \frac{1}{2\pi t} \sqrt{4t - x^2} \mathbf{1}_{[-2\sqrt{t}, 2\sqrt{t}]}(x) dx$$

is the semicircular density.

Moreover, the increments of the process are freely independent, that is for $t_1 < t_2 < \dots < t_n$,

$$\varphi [P_1(X(t_2) - X(t_1)) \dots P_{n-1}(X(t_n) - X(t_{n-1}))] = 0,$$

where P_i 's are polynomials such that

$$\begin{aligned} \varphi [P_1(X(t_2) - X(t_1))] &= \dots \\ &= \varphi [P_{n-1}(X(t_n) - X(t_{n-1}))] = 0. \end{aligned}$$

Therefore $\{X(t)\} =$ free Brownian motion.

The von Neumann algebra generated by $\{X(t)\}$ is $W^*(\mathbb{F}_\infty)$.

Two more things one can do:

For any $q \in [-1, 1]$, define the operator P_q by

$$P_q(\xi_1 \otimes \dots \otimes \xi_n) = \sum_{\sigma \in \text{Sym}(n)} q^{i(\sigma)} \xi_{\sigma(1)} \otimes \dots \otimes \xi_{\sigma(n)},$$

For $q \in (-1, 1)$, P_q is positive definite.

Define the inner product $\langle \bar{\xi}, \bar{\eta} \rangle_q = \langle \bar{\xi}, P_q \bar{\eta} \rangle_0$ and the q -deformed full Fock space $\mathcal{F}_q(\hat{H}_{\mathbb{C}})$ as the completion of $\mathcal{F}_{\text{alg}}(H_{\mathbb{C}})$ with respect to the corresponding norm.

For $\xi \in H$, define creation operators as before, and annihilation operators by

$$\begin{aligned}
 a(\xi)(\eta_1 \otimes \dots \otimes \eta_n) \\
 = \sum_{k=1}^n q^{k-1} \langle \xi, \eta_k \rangle \eta_1 \otimes \dots \otimes \hat{\eta}_k \otimes \dots \otimes \eta_n.
 \end{aligned}$$

Again $a(\xi)$ and $a^*(\xi)$ are adjoints of each other.

Commutation relations:

$$a(\xi)a^*(\eta) - qa^*(\eta)a(\xi) = \langle \xi, \eta \rangle \text{Id}$$

and that's it.

$X(t)$ is the q -Brownian motion.

Another thing one can do: to get other processes, add a “differential second quantization” part (preservation operator).

Simplest example: Poisson process

$$X(t) = a^*(\mathbf{1}_{[0,t)}) + a(\mathbf{1}_{[0,t)}) + p(\mathbf{1}_{[0,t)}),$$

where

$$\begin{aligned} p(f)(g_1 \otimes g_2 \otimes \dots \otimes g_n) \\ = \sum_{k=1}^n q^{k-1} (fg_k) \otimes g_1 \otimes \dots \otimes \hat{g}_k \otimes \dots \otimes g_n \end{aligned}$$

In this way, can get all Lévy processes with all moments.