QUANTIZATION OF SYMPLECTIC REDUCTION

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ABSTRACT. Symplectic reduction, also known as Marsden-Weinstein reduction, is an important construction in Poisson geometry. Following N.P. Landsman [22], we propose a quantization of this procedure by means of M. Rieffel's theory of induced representations. Here to an equivariant momentum map there corresponds an operator-valued rigged inner product. We define the operator-algebraic notions that are involved in this construction, and give a number of examples.

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Added in print: (1) The contemporary name for the structure with the rigged inner product is a Hilbert module. (2) Some related articles are: *Survey of Strict Deformation Quantization* by Bina Bhattacharyya (written for the same course, `http://math.berkeley.edu/~alanw/242papers.html`), and *Morita Equivalence in Algebra and Geometry* by Ralf Meyer, *Von Neumann Algebras and Poisson Manifolds* by Dimitri Shlyakhtenko, both written for Prof. Weinstein's course *Geometric Models for Noncommutative Algebras*, `http://math.berkeley.edu/~aweinst/277papers/277papers.html`.

1. Introduction

Symplectic reduction [1, 12, 28] (also known as Marsden-Weinstein reduction) is one of the basic constructions in symplectic geometry. Given a Lie group G and a hamiltonian action of G on a symplectic manifold S, one gets a momentum mapping $J : S \to \mathfrak{g}^*$ from S to the dual of the Lie algebra of G. If the action is strongly hamiltonian, the momentum map is G-equivariant (here G acts on $\mathfrak{g}^*$ by the coadjoint action). In this case given a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ one can obtain the reduced manifold $J^{-1}(\mathcal{O})/G$ which has a natural symplectic structure induced from $S \times \mathcal{O}$ (cf. the end of Section 2).

In the standard transition from the classical to the quantum mechanical picture ([1], see also [9] for a somewhat different approach) one replaces the symplectic manifolds S and $\mathcal{O}$ by Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_{\mathcal{O}}$, and actions of G on S and $\mathcal{O}$ by unitary representations of G on $\mathcal{H}_S$ and $\mathcal{H}_{\mathcal{O}}$ (more precisely, one considers projective Hilbert spaces $\mathcal{PH}$, i.e. vectors are defined only up to phase=(complex factor of modulus 1)). One usually takes the representation on $\mathcal{H}_{\mathcal{O}}$ to be irreducible. One motivation for this is that since the coadjoint representation of G on $\mathcal{O}$ is transitive, the only G-invariant functions on $\mathcal{O}$, i.e. functions commuting with the action of G, are constant, cf. Example 3. Another is that by a theorem of Kirillov [17], for G nilpotent its space of irreducible representations is actually homeomorphic to the space of coadjoint orbits of $\mathfrak{g}^*$ (both with appropriate topologies).

We will consider a slightly different setting which will allow us to quantize Poisson manifolds as well as symplectic manifolds. Namely, the definition of the Poisson manifold suggests that the main object one works with is not the manifold itself but the Poisson algebra of functions $C^\infty(P)$ on it. Correspondingly, in the quantum case it is natural to consider not the Hilbert space $\mathcal{H}$, to which corresponds the whole algebra $\mathcal{B}(\mathcal{H})$ of bounded operators, but certain subalgebras of $\mathcal{B}(\mathcal{H})$. Two natural classes of such subalgebras are the following [3, 6, 7, 14]:

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Definition 1. An operator *-algebra $A \subset \mathcal{B}(\mathcal{H})$ is a C^**-algebra* if it is closed in the topology induced by the operator norm on $\mathcal{B}(H)$. (An algebra is a **-algebra* if it is closed under the adjoint operation.)

Definition 2. An operator *-algebra $A \subset \mathcal{B}(\mathcal{H})$ is a *von Neumann algebra* (or a W^**-algebra*) if it is closed in the strong operator topology (i.e. the weak topology induced by the linear functionals $a \mapsto \|a\xi\|, \forall \xi \in \mathcal{H}$).

Example 1. Von Neumann algebras (in particular C^*-algebras): $\mathcal{B}(\mathcal{H})$, in particular $M_n(\mathbb{C})$, their direct sums and tensor products; group von Neumann algebras (to be defined below).

Example 2. C^*-algebras that are not von Neumann algebras: algebra of compact operators.

See also Theorems 2 and 3.

The advantage of considering operator algebras other than $\mathcal{B}(\mathcal{H})$ is that once they are defined they can be considered without reference to any particular representation on a Hilbert space (in fact it is possible to define C^*- and W^*-algebras without reference to any particular $\mathcal{B}(\mathcal{H})$). One usually takes quantum algebras of observables to be C^*-algebras (Haag-Kastler [13], cf. also [3]), but since the main object below is the reduced rather than the enveloping group C^*-algebra we might as well restrict our attention to von Neumann algebras. While the constructions below remain valid in the C^*-algebraic case, the W^* case is somewhat more conceptual, mainly because of the following powerful characterization [3, 7, 14]:

Theorem 1 (von Neumann bicommutant theorem)**.** *For a *-algebra $A \subset \mathcal{B}(\mathcal{H})$ containing the identity of $\mathcal{B}(\mathcal{H})$, A is a von Neumann algebra iff $A'' = A$.*

Here A', the *commutant* of A, is the algebra of all operators in $\mathcal{B}(\mathcal{H})$ commuting with all operators in A.

Corollary and Example 3. If a von Neumann algebra $A \subset \mathcal{B}(\mathcal{H})$ and the representation of A on $\mathcal{H}$ is irreducible, the space of the intertwining operators from the representation to itself, which is precisely the commutant of the representation A, is reduced to the scalars, and so $A = A'' = \mathcal{B}(\mathcal{H})$. Thus for an irreducible representation there is no difference between considering the von Neumann algebra or the Hilbert space itself.

The difference between C^*- and von Neumann algebras should be clear from the following characterization of the commutative case (see e.g. [5]):

Theorem 2. *A commutative C^*-algebra with identity is isometrically *-isomorphic to the algebra $C(X)$ of continuous functions on some compact space X. Note that $C(X)$ is an algebra under pointwise multiplication.*

Theorem 3. *A commutative von Neumann algebra (with identity) on a separable (i.e. countably-dimensional) Hilbert space is (spacially) isomorphic to the algebra $L^\infty(X, \mu)$ of bounded measurable functions on a compact metric space X, where μ is a positive Borel measure with support X.*

Thus intuitively C^*-algebras correspond to algebras of continuous functions while von Neumann algebras correspond to algebras of measurable functions. One of the important issues in quantization is that we want to quantize not just "the space of functions on a manifold", but a specific class of them, e.g. smooth functions. By considering von Neumann algebras we avoid this issue completely, and it is not surprising that the arguments simplify.

2. Generalized Symplectic Reduction

Thus, to get back to the construction, we quantize the classical situation

$$\mathcal{O} \hookrightarrow \mathfrak{g}^* \xleftarrow{J} S$$

all with G actions which are assumed to be equivariant, as follows:

$$\mathcal{B}(\mathcal{H}_{\mathcal{O}}) \longleftarrow B_{\mathfrak{g}^*} \longrightarrow \mathcal{B}(\mathcal{H}_S)$$

with G actions on all three spaces. We will suggest a plausible candidate for $B_{\mathfrak{g}^*}$ in Section 4 below. For the moment we note that given this data the symplectic reduction procedure produces a symplectic manifold $J^{-1}(\mathcal{O})/G$ and any function f in $C^\infty(S)$ that is G-invariant projects down to a function in $C^\infty(J^{-1}(\mathcal{O})/G)$. (We will assume that the manifold S is connected and consider the equivalent requirement that f Poisson commute with $J^*C^\infty(\mathfrak{g}^*)$). The quantization of this data is that for any subalgebra $A \subset \mathcal{B}(\mathcal{H}_S)$ commuting with $B_{\mathfrak{g}^*}$, given a representation $B \to \mathcal{B}(\mathcal{H}_\mathcal{O})$ one obtains a representation $A \to \mathcal{B}(\mathcal{H}_{J^{-1}(\mathcal{O})/G})$; representations of B induce representations of A. Note that the quantum construction does not involve the G-actions or the fact that B comes from a particular manifold $\mathfrak{g}^*$.

Now taking the quantum case as a motivation and translating back to the classical situation, we would like to have the following more general reduction procedure: given two classical representations $S_\rho \xrightarrow{\rho} P \xleftarrow{J} S$, where S_ρ and S are symplectic and P is a Poisson manifold, to get a symplectic manifold S^ρ carrying a representation of the subalgebra of $C^\infty(S)$ of functions Poisson commuting with $J^*C^\infty(P)$. Under quite general conditions this has in fact been achieved by Xu [40, Prop. 2.1] and, in the form below, by Landsman [22, Thm.2].

Definition 3. Let S, S_ρ be two connected symplectic manifolds, P a Poisson manifold. Let $J : S \to P^-$ and $\rho : S_\rho \to P$ be two classical representations, i.e. Poisson maps. Then $S{*_P}S_\rho \subset S \times S_\rho$ is defined by

$$S{*_P}S_\rho = \{(x, y) \in S \times S_\rho | J(x) = \rho(y)\}$$

Thus $S{*_P}S_\rho$ completes the diagram
$$\begin{array}{ccc} S{*_P}S_\rho & \longrightarrow & S_\rho \\ \downarrow & & \downarrow \rho \\ S & \xrightarrow{J} & P \end{array}$$
Given $f \in C^\infty(P)$, we also define a vector field $\hat{X}_f$ on $S \times S_\rho$ by $\hat{X}_f(g) = \{J^*f - \rho^*f, g\}$, using the product symplectic structure on $S \times S_\rho$.

Remark 1. Note that $(\rho^*f - J^*f)(x, y) = f(\rho(y)) - f(J(x))$, and so $S{*_P}S_\rho$ is precisely the zero manifold of all such functions.

Remark 2. If P is a linear Poisson manifold, for example a dual of a Lie algebra, then we can define a momentum map $\Phi : S^- \times S^\rho \to P$ by $\Phi = \rho - J$, composed with the appropriate projections. In this case $S{*_P}S_\rho \subset S \times S_\rho$ is just $\Phi^{-1}(0)$ and $\hat{X}_f$ is the hamiltonian vector field of Φ^*f.

Theorem 4. *$S{*_P}S_\rho$ is an immersed coisotropic submanifold of $S \times S_\rho$. The collection of vector fields $\{\hat{X}_f | f \in C^\infty(P)\}$ defines a (generally singular) foliation $\mathcal{F}$ of $S{*_P}S_\rho$ whose leaf space $S^\rho = (S{*_P}S_\rho)/\mathcal{F}$ coincides with the quotient of $S{*_P}S_\rho$ by its characteristic foliation.*

Proof. Denote $M = S{*_P}S_\rho$ for convenience. We show that $\mathcal{F}$ is, on one hand, the singular foliation for M and, on the other hand, is tangent to it. For $X \in T_xS$ and $Y \in T_yS_\rho$, we have $X + Y \in T_{(x,y)}M$ iff $J_*X = \rho_*Y$. The dimension of $T_{(x,y)}M$ at any point (x, y) equals $(\dim S{+}\dim S_\rho - (\mathrm{rank}J_*)(x))$. Therefore the dimension of $(T_{(x,y)}M)^\perp$ in $T(S \times S_\rho)$ is $(\mathrm{rank}J_*)(x)$. Let $\mathcal{F}_{(x,y)}$ be the linear span of all the vector fields $\{\hat{X}_f | f \in C^\infty(P)\}$ at the point (x, y). Then $\dim\mathcal{F}_{(x,y)}$ is $(\mathrm{rank}J_*)(x)$ as well. Therefore to show that $\mathcal{F}_{(x,y)} = (T_{(x,y)}M)^\perp$, it suffices to show $\mathcal{F}_{(x,y)} \subset (T_{(x,y)}M)^\perp$. Indeed, for $X + Y \in T_{(x,y)}M$ as above and $\omega = \omega_S + \omega_{S_\rho}$ the symplectic form on $S \times S_\rho$, we have

$$\omega(X + Y, \hat{X}_f)_{(x,y)} = (d(J^*f - \rho^*f)(X + Y))_{(x,y)} = 0$$

Moreover, $\mathcal{F}_{(x,y)} \subset (T_{(x,y)}M)$ as well: if X_g is the hamiltonian vector field of a function g, then by Lemma 1.2 in [38] $J_*X_{J^*f} = -X_f$, where X_f is defined w.r.t. the Poisson bracket on P (rather than P^-, hence the sign), and $\rho_*X_{\rho^*f} = X_f$. Thus $\hat{X}_f = X_{J^*f} - X_{\rho^*f} \in TM$. Therefore $S{*_P}S_\rho$ is an immersed coisotropic submanifold in $S \times S_\rho$. Furthermore, $[\hat{X}_f, \hat{X}_g] = -\hat{X}_{\{f,g\}}$ (using the

Poisson structure on P), so by the Stefan-Sussmann theorems (cf. [25, Thm. 3.9, 3.10, App. 3]) the distribution $\mathcal{F}$ defines a (singular) foliation. □

Remark 3. Extra regularity conditions are required to guarantee that S^ρ be a manifold.

Corollary 1. *Let $A \subset (J^* C^\infty(P))'$ be a Poisson subalgebra of $C^\infty(S)$. Then the map $\pi^\rho : A \to C^\infty(S^\rho)$, defined by*

$$\pi^\rho(f)(x, y) = f(x)$$

is well defined, and is a Poisson map. Here (x, y) is the equivalence class of (x, y) in S^ρ, and prime denotes the Poisson commutant.

Remark 4. In many examples, $A = \theta^* C^\infty(Q)$ for some classical representation $S \xrightarrow{\theta} Q$, Q Poisson, in which case we get a classical representation $S^\rho \to Q$. Thus given the data

$$Q \xleftarrow{\theta} S \xrightarrow{J} P^-,\ S_\rho \xrightarrow{\rho} P$$

we obtain a new representation

$$S^\rho \longrightarrow Q$$

cf. Section 3 and Definition 6.

In the Marsden-Weinstein case of a group action, we have $P = \mathfrak{g}^*$, a dual of a Lie algebra and $S_\rho = \mathcal{O}$, a coadjoint orbit, with ρ the inclusion map. The property of P implies that we have the product momentum map $\Phi : S^- \times \mathcal{O} \to \mathfrak{g}^*$, and the fact that the inclusion map is injective implies that the projection of $S *_P \mathcal{O}$ onto $J^{-1}(\mathcal{O}) \subset S$ is injective. Therefore in this case we may consider $S *_P \mathcal{O}$ as a submanifold of S, and the reduced manifold as its quotient. Note, however, that while $J^{-1}(\mathcal{O})$ is a coisotropic submanifold of S, its characteristic foliation is different from its G-foliation, while they do coincide for $S *_P \mathcal{O} \subset S^- \times \mathcal{O}$ (as in the Theorem). Thus the isomorphism $S \times^*_{\mathfrak{g}} \mathcal{O}/\mathcal{F} \cong J^{-1}(\mathcal{O})/G$ is an efficient way of providing $J^{-1}(\mathcal{O})/G$ with its correct symplectic structure (the usual way of defining it is by the isomorphism $J^{-1}(\mathcal{O})/G \cong J^{-1}(\mu)/G_\mu$ for $\mu \in \mathcal{O}$).

3. Rieffel Induction

This construction is a possible quantum counterpart of the generalized Marsden-Weinstein reduction. One is given two algebras A and B and a vector space L which is a left A- module and a right B-module, $A \to \mathrm{End}(L) \leftarrow B$. A and B are usually taken to be C^*-algebras, but in some examples only weaker conditions are satisfied and one has to adjust the construction accordingly. L is called a *Hilbert (C^*-) bimodule* [18]. Given a representation π_χ of B on a Hilbert space $\mathcal{H}_\chi$, we want to induce a representation of A on some other Hilbert space $\mathcal{H}^\chi$. For example, if B is a subalgebra of A and $L = A$, we want to know which representations of an algebra come from those of a subalgebra. The corresponding problem for groups has been considered by G. Frobenius [10] and, in a more functional-analytic context, by G. Mackey (see e.g. [26, 27] and references in [32]). M. Rieffel found the following general construction [32]. The basic idea probably goes back to Stinespring [36].

Suppose we are given on L a *rigging map*, i.e. a B-valued inner product $\langle \cdot, \cdot \rangle_B$. This is a linear map $L \times L \to B$ satisfying the following conditions for all $\psi, \phi \in L$:

- **(1):** $\langle \lambda\psi, \mu\phi \rangle_B = \overline{\lambda}\mu \langle \psi, \phi \rangle_B \ \forall \lambda, \mu \in \mathbb{C}$ ($\mathbb{C}$-sesquilinearity)
- **(2):** ${\langle \psi, \phi \rangle_B}^* = \langle \phi, \psi \rangle_B$ (B- sesquilinearity)
- **(3):** $\langle \psi, \phi b \rangle_B = \langle \psi, \phi \rangle_B \, b \ \forall b \in B$ (connection with the B-action)
- **(4):** $\langle a\psi, \phi \rangle_B = \langle \psi, a^* \phi \rangle_B \ \forall a \in A$ (connection with the A-action)

We will see examples of such maps below. Given a rigging map and a representation of B on $\mathcal{H}_\chi$, we can form the (say, algebraic) tensor product $L \otimes \mathcal{H}_\chi$. If the rigging map is positive, i.e. $\forall \psi \in L, \langle \psi, \psi \rangle_B \geq 0$ is a positive operator in B, or at least that π_χ is L-positive in the sense that

$\forall \psi \in L, \pi_\chi(\langle \psi, \psi \rangle_B) \geq 0$ is a positive operator on $\mathcal{H}_\chi$, then the following bilinear form on $L \otimes \mathcal{H}_\chi$ is positive semi- definite:

$$(\psi \otimes v, \phi \otimes w)_0 = (\pi_\chi(\langle \phi, \psi \rangle_B)v, w)_{\mathcal{H}_\chi}$$

where $(\cdot,\cdot)_{\mathcal{H}_\chi}$ is the inner product on $\mathcal{H}_\chi$.

Then given an A-B-bimodule L with a rigging map on it and a representation of B on $\mathcal{H}_\chi$, we construct a representation of A as follows:

Step 1: Form the algebraic tensor product $L \otimes \mathcal{H}_\chi$ with a positive semi- definite bilinear form on it, as above.

Step 2: Form the tensor product over B $L \otimes_B \mathcal{H}_\chi$. This is possible since L is a right B-module and $\mathcal{H}_\chi$ is a left B-module. The bilinear form projects to this tensor product by condition (3) in the definition of the rigging map.

Step 3: Quotient $L \otimes_B \mathcal{H}_\chi$ out by the subspace of elements of norm zero (where the norm is induced by the above bilinear map). We obtain a vector space with a positive definite bilinear form. After completing with respect to that form we obtain a Hilbert space $\mathcal{H}^\chi$. The (left) representation of A on L carries over to the space $\mathcal{H}^\chi$, because of the conditions (3) and (4). If A is a C^*-algebra, this representation is actually bounded. If A is not complete in the C^*-norm, we need an extra boundedness condition on the rigging map and the representation π_χ.

We have the following analogy with Theorem 4 (see especially Remark 4). The momentum map $J : S \to P^-$ is both a map between spaces and induces a Lie algebra antihomomorphism $J^* : C^\infty(P) \to C^\infty(S)$. It corresponds both to the rigging map $L \times L \to B$ and the antirepresentation $\mathrm{End}(L) \leftarrow B$. Then given a representation $S^\rho \to P$, we (1) form the direct product $S \times S^\rho$ (3) take a coisotropic submanifold $S *_P S^\rho$ and (2) quotient out by the foliation generated by the hamiltonian vector fields of the pullbacks of functions on P. The analogy is not exact. First, in the algebraic approach the regularity questions (whether S_ρ is a manifold) are missing. Second, in the quantum case Step 2 (product over B) is not really necessary: it is implied by Step 3. This is certainly not true in the commutative case. Third, and perhaps most important, unlike in the commutative case, the existence of a representation by no means implies the existence of a rigging map. In fact in specific examples (i.e. in physics) finding the rigging map is the main issue. We will see, however, that the analogy is very close in the original Marsden-Weinstein/Mackey case, i.e. for group actions.

We consider a sequence of examples of increasing complexity.

4. The Group Algebra

The simplest example of symplectic reduction is the following: for a Lie group G, the action of G on itself by right translation induces a strongly hamiltonian action of G on T^*G, with the momentum map $J : T^*G \to \mathfrak{g}^{*-}$ being just the fiberwise dual of the vertical inclusion maps $\mathfrak{g} \to TG$. $J^*C^\infty(\mathfrak{g}^{*-})$ are the hamiltonian functions of the left-invariant hamiltonian vector fields, i.e. the left- invariant functions. Their Poisson commutant is precisely the right invariant functions, thus in this case we have the dual Poisson manifold $T^*G/G = \mathfrak{g}^*$, with the dual momentum map $J^- : T^*G \to \mathfrak{g}^*$ induced by the left action of G (see Definition 6). The representation of $C^\infty(\mathfrak{g}^*)$ on a coadjoint orbit $\mathcal{O}$ induces again the representation of $C^\infty(\mathfrak{g}^*)$ on $\mathcal{O}$.

We claim that the appropriate quantum analog of $\mathfrak{g}^*$ is the group von Neumann algebra $W^*(G)$ [5, 7, 14].

Definition and Construction 4. Let G be a locally compact (for example discrete) group. We have a (left) Haar measure dx on G which to simplify notation, is assumed to be unimodular. We can define the Hilbert space $L^2(G, dx)$. G acts on itself, and hence on $L^2(G)$, by left translation, thus inducing the left regular representation $g \mapsto U_g$ of G on $L^2(G)$. We extend this representation to functions on G by using the following *integrated form* σ:

for $x, y \in G$, $\xi \in L^2(G)$, $f, g \in L^1(G)$

$$(U_y \xi)(x) = \xi(y^{-1}x)$$

$$(\sigma(f)\xi)(x) = (\int_G f(y)(U_y\xi)dy)(x) = \int_G f(y)\xi(y^{-1}x)dy$$

To make this a *-homomorphism, we are forced to define the convolution multiplication on $L^1(G)$ by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$$

and the involution by

$$f^*(x) = \overline{f(x^{-1})}$$

Then the above representation on $L^2(G)$ is a faithful *-representation. The W^*- completion of $L^1(G)$ in this representation is called the group von Neumann algebra of G. One of the simplifications of the W^*-approach is that the result is the same if we start with functions of compact or even finite support on G. In particular, $U_g \in W^*(G)$. Note also that in general a function from a set X into a set Y can be thought of as a formal linear combination of elements of X with coefficients in Y.

We claim that $W^*(G)$ is the appropriate quantization of $C^\infty(\mathfrak{g}^*)$. Very superficially, we can say that the convolution algebra of functions on G approximates the convolution algebra of functions on $\mathfrak{g}$ (and vice versa), which in turn is equal to the algebra of functions on $\mathfrak{g}^*$ under pointwise multiplication (the Fourier transform takes convolution to multiplication and moreover takes the L^1 norm induced from the left regular representation to the L^∞ norm). This statement can be made much more formal.

Among the numerous approaches to the problem of quantization one of the most common is the (formal) deformation quantization, introduced by Bayen et al. [2]. Given a Poisson algebra A, one considers a family of associative algebras A_h, $A_0 = A$ indexed by a real parameter h (Plank's constant). The underlying space of A_h is the space of formal power series in h with coefficients in A. The multiplication maps $*_h$ are required to have the following properties: for $a, b \in A$

$$a *_h b = ab + O(h)$$

$$-\frac{i}{h}(a *_h b - b *_h a) = \{a, b\} + O(h)$$

The construction is completely formal and does not treat the questions of convergence and topology in general. To rectify this Rieffel has introduced the concept of a strict deformation quantization [33]. Very briefly, sometimes one can include all the algebras A_h in a single C^*-algebra so that the multiplications are all induced from the ambient algebra. Then we get consistent topologies on the whole family A_h.

The construction of the deformation quantization of $\mathfrak{g}^*$ (i.e. of the algebra of C^∞ functions on it) goes as follows [34]: for a parameter h we define a new bracket on $\mathfrak{g}$: $[X, Y]_h = h[X, Y]$. This defines a family of Lie algebras $\mathfrak{g}_h$ with $\mathfrak{g}$ as the underlying space; $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_0$ abelian. By exponentiating we obtain a family of simply connected Lie groups G_h; $G_1 = G$, $G_0 = \mathfrak{g}$ abelian. Note that the above description of A_h is very similar to the way one obtains a Lie group from a Lie algebra. At least locally, we can transfer the convolution multiplication from G_h to $\mathfrak{g}_h$ via the exponential map, and then to a deformed product $*_h$ on $\mathfrak{g}^*$ via the Fourier transform scaled by h. This does in fact give a deformation quantization of $\mathfrak{g}^*$, which is even strict in the case when $\mathfrak{g}$ is nilpotent. Of course, lots of important details are missing in this exposition.

The above discussion was supposed to demonstrate that the appropriate quantum analog of $\mathfrak{g}^*$ is the group algebra $W^*(G)$. Accordingly, the quantization of the data

$$\mathfrak{g}^* \xleftarrow{J^-} T^*G \xrightarrow{J} \mathfrak{g}^{*-},\ \mathcal{O} \hookrightarrow \mathfrak{g}^*$$

is

$$W^*(G) \longrightarrow \mathcal{B}(L^2(G)) \longleftarrow W^*(G),\ W^*(G) \to \mathcal{B}(\mathcal{H}_{\mathcal{O}})$$

The analog of the statement that the Poisson commutant of the space of left-invariant functions are the right-invariant functions is that the commutant of the left regular representation is precisely the right regular anti-representation.

We can apply Rieffel's induction procedure to this situation. The rigging map on the dense subspace $L^1(G) \cap L^2(G)$ of $L^2(G)$ with values in $L^1(G) \subset W^*(G)$ is just the convolution product. We form $L^2(G) \otimes \mathcal{H}_{\mathcal{O}}$, with norm

$$\|f \otimes \eta\|^2 = ((f,f)_{W^*(G)}\eta, \eta)_{\mathcal{H}_{\mathcal{O}}} = ((f^* * f)\eta, \eta) \geq 0$$

The norm is equal to 0 iff f is in the kernel of the representation on $H_{\mathcal{O}}$. Thus the induced representation we obtain is the same representation, just like in the Poisson case.

5. Crossed Products

The next simplest situation is the action on T^*G of a subgroup $H \leq G$.

$$T^*G/H \longleftarrow T^*G \xrightarrow{J} \mathfrak{h}^{*-}$$

$$C^\infty(T^*G)^H = C^\infty(T^*G/H) \longrightarrow C^\infty(T^*G) \xleftarrow{J^*} C^\infty(\mathfrak{h}^{*-})$$

The appropriate operator algebraic notion in this case is one of a crossed (or semidirect) product [3, 7, 14, 30]. Suppose we are given an algebra $A \subset \mathcal{B}(\mathcal{H})$ and a group G acting on it by an action β. Form the Hilbert space $L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$. We have representations π of A and U of G (by representations of a group we mean unitary representations) on this space as follows:

$x, y \in G,\ a \in A,\ \xi \in L^2(G, \mathcal{H}),\ f, g \in L^1(G, A)$

$$(U_y\xi)(x) = \xi(y^{-1}x)$$

$$(\pi(a)\xi)(x) = \beta_{x^{-1}}(a)(\xi(x))$$

Thus the representation of G is just the left regular representation, but the representation of A is twisted by β. The purpose of this is that

$$\begin{aligned} ((U_y\pi(a){U_y}^{-1})\xi)(x) &= (\pi(a){U_y}^{-1})\xi(y^{-1}x) = \beta_{x^{-1}y}(a)(({U_y}^{-1}\xi)(y^{-1}x)) \\ &= \beta_{x^{-1}}(\beta_y(a))\xi(x) = (\pi(\beta_y(a))\xi)(x) \end{aligned}$$

$$U_y\pi(a){U_y}^{-1} = \pi(\beta_y(a))$$

Thus in this representation the β-action is just the conjugation action. Again we construct the integrated form representation

$$(\sigma(f)\xi)(x) = (\int_G \pi(f(y))U_y\xi dy)(x) = \int_G \pi(\beta_{x^{-1}}(f(y))\xi(y^{-1}x)dy$$

and in order to make this a *-representation have to define

$$(f *_\beta g)(x) = \int_G f(y)\beta_y(g(y^{-1}x))dy$$

$$f^{*_\beta}(x) = \beta_x(f(x^{-1})^*)$$

Taking the von Neumann closure in this representation we obtain the crossed product algebra $A \times_\beta G$. In the case when A is a commutative algebra, $A \times_\beta G$ is also called the *transformation group algebra*. The connection with the semidirect products is that if $A = W^*(K)$ (so G acts on K by β) and the group H is the semidirect product $K \times_\beta G$, then

$$A \times_\beta G = W^*(K) \times_\beta G = W^*(K \times_\beta G) = W^*(H)$$

Consider the crossed product $C_0(G/H) \times_\beta G$, where G is a locally compact group, H a subgroup, and the action of G on the homogeneous space G/H is induced from the right action of G on itself. For $H = G$, this is just $W^*(G)$, which as we saw above, is a quantization of $C^\infty(\mathfrak{g}^*)$. By methods very similar to Rieffel's approach above (which in turn has been inspired by the original quantum mechanical Weyl quantization of $T^*\mathbb{R}^n$ using the Moyal product [9], cf. Example 4) Landsman shows in [19] that such a crossed product is naturally a deformation quantization of the Poisson algebra $C(T^*G/H)$. It is not hard to show that the Poisson manifolds $\mathfrak{h}^*$ (given by the left action of H on G) and T^*G/H are dual Poisson manifolds in T^*G. On the other hand, the representations of $W^*(H)$ (again given by the left action of H on G) and the crossed product $C_0(G/H) \times_\beta G$ coming from the right action of G on G on the Hilbert space $L^2(G)$ are precisely the commutants of each other.

6. The Universal Phase Space

Now we consider the action of a Lie group H on the cotangent bundle of an arbitrary manifold P (P need not be Poisson). Actually, from the physical point of view it is natural to assume that P is a principal H-bundle over some manifold Q. In this case H acts on P (and hence on T^*P) in a natural way. Note also that if P itself is a Lie group we get the situation in the previous section.

Thus the picture is

$$T^*P/H \longleftarrow T^*P \xrightarrow{J} \mathfrak{h}^{*-}$$

We want to quantize T^*P/H. It follows from Landsman's arguments in [21] (applied to the von Neumann algebra context) that such a quantization is $W^*(H) \otimes \mathcal{B}(L^2(Q))$ (the action of H on H being the right action). In fact it is easy to see that this is the commutant of the left representation of $W^*(H)$ on $L^2(P)$. Indeed, while P is only a vector bundle over Q, a choice of a (measurable) cross-section gives a (non-canonical) splitting $L^2(P) = L^2(H) \otimes L^2(Q)$. The representation of $W^*(H)$ on this tensor product is (left regular representation on $L^2(H)$)$\otimes$(identity on $L^2(Q)$. By an important (and very non- trivial) theorem [16, Ch. 13], a commutant of a tensor product is a tensor product of commutants. Therefore the commutant in this case is (right regular representation on $L^2(H)$)$\otimes \mathcal{B}(L^2(Q))$, as stated.

Remark 5. This fact can also be shown by elementary means using the Hilbert-Schmidt integral operators in $L^2(P \times P)$ [21, Sec.3].

The group action of a group H on a cotangent bundle of a principal H-bundle P with base space Q has a very natural physical meaning. It was first pointed out by Sternberg [35] and refined in [29, 37]. For $H = U(1)$, the reduction at a coadjoint orbit $\mathcal{O} = \{e\} =$ (point) corresponds to the description of the motion of a particle with charge $\{e\}$ in the electromagnetic field. More generally, for arbitrary H one thinks of the interaction of the Yang-Mills field with the gauge group H with a particle with "charge" = coadjoint orbit of H. An important point here is that while P, and hence T^*P and the reduced manifold $J^{-1}(\mathcal{O})/H$ are bundles over Q, they are not naturally bundles over T^*Q. However, $J^{-1}(\mathcal{O})/H$ and hence T^*P do become bundles over T^*Q given a choice of a connection on P.

As stated above, the main obstacle to applying Rieffel's induction procedure is finding the operator-valued inner product on the given bimodule. In the classical situation, while one does not always obtain a dual pair of Poisson manifolds given just a map $S \to P$, one has the reduction procedure in the case where $P = \mathfrak{g}^*$, under mild conditions on the action, by the standard Marsden-Weinstein reduction. We now show that in the operator algebraic case if the algebra one is inducing from is a group algebra of a locally compact group, then the induction procedure goes through [32]. The main point is that given a representation π of $W^*(G)$ on a (pre-)Hilbert space (rather than just a vector space) $\mathcal{H}$, one can define a $W^*(G)$-valued inner product on $\mathcal{H}$. Indeed for $\psi, \phi \in \mathcal{H}$ one defines $\langle \psi, \phi \rangle_{W^*(G)}$ to be a function on G as follows: $g \mapsto (\pi(g)\phi, \psi)_{\mathcal{H}}$. Such a function is in $L^1(G)$ and therefore is an element of $W^*(G)$. One also shows that this rigging map is positive if G is compact, and L-positive (i.e. the induced inner product on the tensor product is positive) if the representation one is inducing from is weakly contained in the left regular representation of G (all representations of G have this property precisely when G is amenable).

7. Projective Representations

In this section we consider the generalization of Marsden-Weinstein reduction in a somewhat different direction. Namely, we assume that the action of the group G on the symplectic manifold S is hamiltonian but not necessarily strongly hamiltonian. That is, we still obtain the momentum map $J : S \to \mathfrak{g}^{*-}$ and the corresponding map $J^{*-} : \mathfrak{g} \to C^\infty(S^-)$, but the latter map is no longer required to be a Lie algebra homomorphism. It is well known how to deal with this situation in the classical case ([12, 28]). If $\{J^*(u), J^*(v)\} \neq J^*([u, v])$, $u, v \in \mathfrak{g}$, then the obstruction to the action being strongly hamiltonian is given by the 2-cocycle $\Sigma \in \mathfrak{g}^* \otimes \mathfrak{g}^*$,

$$\begin{aligned} \Sigma(v, u) &= \{J^*(u)(s), J^*(v)(s)\} - J^*([u, v])(s) \\ &= \{\langle J(s), u\rangle, \langle J(s), v\rangle\} - \langle J(s), [u, v]\rangle \end{aligned}$$

which depends on the connected component of $s \in S$ and not on s itself. This cocycle can be eliminated by changing the momentum map if and only if it is a coboundary. However, in general we can define an affine Poisson bracket $\{\cdot, \cdot\}^\Sigma$ on $C^\infty(\mathfrak{g}^*)$ by

$$\{\tilde{u}, \tilde{v}\}^\Sigma = \widetilde{[u, v]} + \Sigma(u, v) 1_{\mathfrak{g}^*}$$

where $\tilde{u} \in C^\infty(\mathfrak{g}^*)$ is defined by $\tilde{u}(\theta) = \langle \theta, u\rangle$, and $1_{\mathfrak{g}^*}$ is the identity function on $\mathfrak{g}^*$. Note that the bracket is thus uniquely defined. Then J is a Poisson map with respect to this modified Poisson structure on $\mathfrak{g}^*$, and is equivariant with respect to the original action of G on S and the modified coadjoint action π_{co}^Σ of G on $\mathfrak{g}^*$ given by

$$\pi_{co}^\Sigma(g)\theta = \pi_{co}(g)(\theta + J(s)) - J(sg^{-1})$$

where π_{co} is the coadjoint action. This is again independent of s if S is connected.

Therefore in order to quantize this situation we want to quantize $\mathfrak{g}^*$ with affine Poisson structure given by a 2-cocycle [22]. The appropriate operator-algebraic notion is one of a *twisted group algebra*. First let the group G be discrete. In this case the group algebra of G is essentially generated by the unitaries $U_x, x \in G$ subject to the relations $U_x U_y = U_{xy}$, $U_{x^{-1}} = {U_x}^{-1}$, $U_1 = 1$. In a twisted algebra we want to modify the multiplicative relation to $U_x U_y = c(x, y) U_{xy}$. The product should be a unitary, and the multiplication should be associative, therefore we have the following conditions on c:

$$|c(x, y)| = 1$$

$$c(x, y)c(xy, z) = c(x, yz)c(y, z)$$

Such a c is called a *multiplier* (no relation to the multiplier algebra). In the quantization we would expect the infinitesimal version of c, i.e. the 2-cocycle on $\mathfrak{g}$ derived from it, to be $-\Sigma$ (and vice versa).

For such c the representation U is called a projective representation, because by a theorem of Wigner any unitary representation on the projective Hilbert space (i.e. all the maps in the image of the representation preserve the absolute values of inner products) lifts to unitary or antiunitary twisted representation of the above form on the Hilbert space itself.

We proceed as in Sections 4 and 5, and use the same notation. We have a projective unitary left regular representation of G on $L^2(G)$

$$(U_y\xi)(x) = c(x^{-1}, y)\xi(y^{-1}x)$$

its integrated form

$$(\sigma(f)\xi)(x) = (\int_G f(y)(U_y\xi)dy)(x) = \int_G f(y)c(x^{-1}, y)\xi(y^{-1}x)$$

and the twisted multiplication and involution on $L^1(G)$

$$(f *_c g)(x) = \int_G f(y)g(y^{-1}x)c(y, y^{-1}x)$$

$$(f^{*_c})(x) = c(x, x^{-1})\overline{f(x^{-1})}$$

The twisted group algebra $W^*(G, c)$ is the completion of this representation on $L^2(G)$. We have the rigged inner product given by the procedure at the end of Section 6.

Example 4 (Weyl Quantization)**.** The oldest example of quantization procedure is Weyl's quantization of $T^*\mathbb{R}^n$. What follows is a somewhat modernized treatment of that construction. See [9] for more explicit approach and [15] for operator-theoretic background; cf. also [31]. For simplicity we put $n = 1$. Thus we consider the symplectic manifold $T^*\mathbb{R} = \mathbb{R}^2$ with $\mathbb{R}^2$ acting on it by translations. The action is not strongly hamiltonian: $J^*(x, y)(\alpha, \beta) = x\alpha - y\beta$, where (x, y) lies in the Lie algebra of $\mathbb{R}^2$ and $(\alpha, \beta) \in T^*\mathbb{R}$. Then while the regular Lie algebra of $\mathbb{R}^2$ is abelian, the new Lie algebra structure on it is given essentially by the symplectic form on $T^*\mathbb{R}$, i.e. by the cocycle

$$\Sigma((x_1, y_1), (x_2, y_2)) = \{J^*(x_1, y_1), J^*(x_2, y_2)\} = x_2y_1 - x_1y_2$$

In other words, this $C^\infty(\mathbb{R}^2)^*$ is generated by two elements x, p subject to the relations

$$[x, x] = 0 = [p, p]$$

$$[p, x] = 1$$

These relations are called *canonical commutation relations* (CCRs). The corresponding multiplier, obtained by "exponentiating" the above relations, is given by $c((a_1x + b_1p), (a_2x + b_2p)) = \exp(i(a_1b_2 - a_2b_1)/2)$. Thus in this case the twisted group algebra is basically generated by the unitaries $U_s,\ s \in \mathbb{R}^2$ subject to the relations $U_sU_t = \exp(i[s, t]/2)U_{s+t}$, or equivalently $U_sU_t = \exp(i[s, t])U_tU_s$, where $[\cdot, \cdot]$ is now the symplectic form on $\mathbb{R}^2$. These are the *Weyl commutation relations.* The twisted C^*-algebra given by these relations is the algebra of compact operators $\mathcal{K}(L^2(G))$, and therefore the twisted von Neumann algebra is $\mathcal{B}(L^2(G))$.

8. Odds and Ends

(1) One of the main results of Rieffel's theory of induced representations is the following

Theorem 5 (Imprimitivity Theorem)**.** [32] *There exists an A-C imprimitivity bimodule L if and only if there is a bijective correspondence between the sets of L-positive representations of A and C. In this case A and C are called* strongly Morita equivalent*.*

Here

Definition 5. L is an A-C *imprimitivity bimodule* if it has both A-valued and C- valued rigged inner products, which satisfy the following extra compatibility conditions:

i) $\mathrm{span}\{\langle\psi,\phi\rangle_C|\psi,\phi\in L\}$ is dense in C, and similarly for A.

ii) ${}_A\langle\psi,\phi\rangle\zeta=\psi\langle\phi,\zeta\rangle_C$ for all $\psi,\phi,\zeta\in L$.

The point is, of course, that using this data we can induce representations of A from those of C and vice versa. Note also that given just the C-valued inner product on L, we can always find an algebra A such that the above conditions are satisfied. Indeed, given any operator- valued inner product $\langle\cdot,\cdot\rangle$ on L, we always have the second one given by "rank one operators": $\langle\langle\psi,\phi\rangle\rangle:\zeta\to\psi\langle\phi,\zeta\rangle$ (see [31] for the simplest case of this construction; cf. also [9, Prop. 1.46]). It will then follow from the properties of the first rigged inner product that the second one is again a rigged inner product, and moreover that their images commute. Note, however, that it is not true in general that the algebra generated by these rank- one operators (called the *imprimitivity algebra* of C) is the commutant of C, or indeed a von Neumann algebra.

The classical counterparts of these notions are the following:

Definition 6. [24, 38, 41] A *classical equivalence bimodule* of a pair of Poisson manifolds (P_1,P_2) consists of a symplectic manifold S and a pair of Poisson morphisms $J_1:S\to {P_1}^-$ and $J_2:S\to P_2$, such that $P_2\xleftarrow{J_2}S\xrightarrow{J_1}{P_1}^-$ is a complete *full dual pair* with connected and simply connected fibers. This means that ${J_1}^*C^\infty({P_1}^-)$ and ${J_2}^*C^\infty(P_2)$ are each other's Poisson commutants in $C^\infty(S)$, that the leaf spaces of the foliations defined by the fibers of J_1 and J_2 are manifolds in the quotient topology, and that J_1 and J_2 are surjective, as well as complete Poisson maps.

In this case P_1 and P_2 are called *Morita equivalent.*

Theorem 6. [22, 39, 40, 41] *Let P_1 and P_2 be Morita equivalent Poisson manifolds. Then there is a bijective correspondence between their respective symplectic realizations.*

Remark 6. To be more precise, the above theorem of Xu corresponds to the case of Rieffel's theorem where the representations are required to be strictly positive, and the above theorem of Rieffel corresponds to the case of Xu's theorem where the foliations of J_1 and J_2 are not required to be manifolds.

(2) In a manner analogous to a group algebra one can introduce an algebra of a groupoid [4, I.1,II.5,V.4]. Indeed, the definition of convolution of two functions on a group G can be written as

$$(f*g)(x)=\int_{y\in G}f(y)g(y^{-1}x)=\int_{yz=x}f(y)g(z)$$

In particular the definition uses only the groupoid property of G, i.e. the products need not be defined for all $y,z\in G$. Under certain discreteness/smoothness conditions on the groupoid one can define the corresponding algebra in an analogous manner. In fact, from the point of view of operator algebras groupoid algebras are more natural, since, for example, under quite general conditions a group transformation von Neumann algebra is determined just by the orbit structure and does not depend on the group action itself. Moreover, the class of groupoid von Neumann algebras is very wide [8]. It includes the group algebras, but it also includes, for example, the matrix algebras. The groupoid giving $M_n(\mathbb{C})$ is just the pair groupoid on the set of n elements.

In [21] Landsman shows that for the Lie algebroid $\mathfrak{L}(\Gamma)$ associated to a Lie groupoid Γ, the quantization of the Poisson algebra $C^\infty(\mathfrak{L}(\Gamma)^*)$ is the groupoid C^*-algebra $C^*(\Gamma)$. Note that this includes quantizations of $C^\infty(\mathfrak{g}^*)$ as well as $C^\infty(T^*M)$. Also, the construction of rigged inner product at the end of Section 6 works perfectly well for groupoids [22].

(3) Section 5 (Crossed Products) is connected with the simplest case of the basic construction of Jones [11]. For a group G and a (say, closed and open) subgroup H, we have the left representation of $W^*(H)$ on $L^2(G)$ as the commutant of the (right) crossed product $C(G/H) \times_\beta G$. Note that $L^2(G)$ is $W^*(G)$ in any faithful GNS representation. The corresponding first step in the Jones tower is $W^*(H) \subset W^*(G) \subset C(G/H) \times_\beta G$, i.e. $C(G/H) \times_\beta G$ is generated by $W^*(G)$ and the orthogonal projection $e_1 : L^2(G) \to L^2(H)$ as operators on $L^2(G)$. Note that the corresponding conditional expectation $E_1 : W^*(G) \to W^*(H)$ is precisely what gives the $W^*(H)$-valued rigged inner product on $L^2(G)$. Indeed, this conditional expectation is $E_1 : f \mapsto f|H$, for $f \in L^1(G)$, and the rigged inner product is given by $\langle f, g\rangle = E_1(g^* * f)$. Note that one of the purposes of Rieffel's construction is to generalize the above situation to, say, subgroups that are closed but not open.

(4) In a sense continuing the previous remark, we can generalize Section 4 (Group Algebra) to arbitrary symplectic groupoids. On the classical side, we use the notation of Theorem 4. For a symplectic groupoid S with base P, we have two Poisson maps $S \xrightarrow{\alpha} P \xleftarrow{\beta^-} S^-$, so in this case $S_\rho = S$. The manifold $S *_P S$ is just the set of multipliable pairs $S \times_{\alpha\beta} S$. There is a natural map from this manifold to S given by the multiplication map.

We show that the foliation given by this map is the characteristic foliation of $S *_P S$. For this we have to use the stronger property of a symplectic groupoid that the graph of the multiplication map $\Gamma_m = \{(g, h, gh)\}$ is a lagrangian submanifold in $S^- \times S^- \times S$. For convenience denote the factors by S_1, S_2, S_3, with projections π_1, π_2, π_3. Note that $\pi_1 \times \pi_2$ is an injective map of Γ_m onto $S_1 *_P S_2$, while the fibers of π_3 are precisely the fibers of m. A vector tangent to Γ_m can be written as $X + Y$, $X \in T(S_1 *_P S_2)$, $Y \in TS_3$. Now for any $\hat{Z}$ as in Theorem 4, $\hat{Z}$ is in the characteristic foliation of $S_1 *_P S_2$, hence $X \perp \hat{Z}$ (symplectically). Also clearly $Y \perp \hat{Z}$. Therefore $\hat{Z} \perp T\Gamma_m$. But Γ_m is lagrangian, hence $\hat{Z} \in T\Gamma_m$. Being also in the fiber of π_3, $\hat{Z}$ has to be tangent to the multiplication foliation.

We also have to show that the reduced manifold S obtained in this way has the correct symplectic structure.

Then taking for A the commutant of $\alpha^* C^\infty(P) \subset C^\infty(S)$, which is precisely $\beta^* C^\infty(P)$, the induced representation will be the same representation on S, just like in the case of T^*G.

In the quantum case, the properties of the maps α, β of a symplectic groupoid correspond to two representations of the same algebra A on some L which are each other's commutants. Moreover, the definition of a symplectic groupoid here apparently corresponds to the requirement that L be a Hilbert algebra (or possibly a left Hilbert algebra). That is, if L is a Hilbert space it is the completion of A in some GNS representation with respect to a faithful state (or weight), while in general vector space case we just take $L = A$ (cf. the group case in the previous remark). Note that this would seem to indicate that for any (quantizable) Poisson manifold there is a symplectic groupoid having it as base. In any case, the rigged inner product here is just the multiplication in L (or A), and the induced representation is the same as the inducing one.

One of the most general cases when there is known a natural rigged inner product seems to be when L is an algebra, $A \subset L$ a subalgebra, and there is a conditional expectation $E : L \to A$. Then the construction is like in the previous comment, and again we can induce representations of the commutant. In this context, it would be interesting to see if there is a useful classical construction generalizing $T^*G/H \leftarrow T^*G \to \mathfrak{h}^{*-}$.

References

[1] S. Bates and A. Weinstein, *Lectures on the Geometry of Quantization*, Berkeley Mathematics Lecture Notes v.8 (1995)

[2] F. Bayen, M. Flato, C. Fronsal, A. Licherowicz and D. Sternheimer, Deformation theory and quantization I, II, Ann. Phys. 110 (1978) 61–110, 111–151
[3] O. Bratelli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics I* (Springer, New York, 1979)
[4] A. Connes, *Noncommutative Geometry* (Academic Press, New York, 1994)
[5] J.B. Conway, *A Course in Functional Analysis*, 2nd ed. (Springer, New York, 1990)
[6] J. Dixmier, *C^*-algebras and Their Representations* (North-Holland, Amsterdam, 1977)
[7] J. Dixmier, *Von Neumann Algebras* (North Holland, Amsterdam, 1981)
[8] J. Feldman and C. Moore, Ergodic equivalence relations, cohomology and von Neumann algebras I, II, Trans. AMS 234 (1977) 289–324, 325–359
[9] G.B. Folland, *Harmonic Analysis in Phase Space* (Princeton University Press, Princeton, 1989)
[10] G. Frobenius, Über relationen zwischen den characteren einen gruppe und denen ihrer untergruppen, Sitzber. Preuss. Akad. Wiss. (1898) 501–515
[11] F. Goodman, P. de la Harpe and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, MSRI Publ. 14 (Springer, New York, 1989)
[12] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge University Press, Cambridge, 1984)
[13] R. Haag and D. Kastler, An algebraic approach to quantum field theory, J. Math Phys. 5 (1964) 848–861
[14] P. de la Harpe and V.F.R. Jones, *An Introduction to C^*-algebras* (to appear)
[15] H. Helson, *The Spectral Theorem*, LNM 1227 (Springer, New York, 1986)
[16] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras II* (Academic Press, New York, 1986)
[17] A.A. Kirillov, Unitary representations of nilpotent Lie groups, Uspehi Mat. Nauk 17 (1962) 57–110
[18] E.C. Lance, *Hilbert C^*-modules* (Cambridge University Press, Cambridge, 1995)
[19] N.P. Landsman, Induced representations, gauge fields, and quantization on homogeneous spaces, Rev. Math. Phys. 4 (1992) 503–527
[20] N.P. Landsman, Deformations of algebras of observables and the classical limit of quantum mechanics, Rev. Math. Phys 5 n.4 (1993) 775–806
[21] N.P. Landsman, Strict deformation quantization of a particle in external gravitational and Yang-Mills fields, J. Geom. Phys. 12 (1993) 1–40
[22] N.P. Landsman, Rieffel induction as generalized quantum Marsden-Weinstein reduction, J. Geom. Phys. 15 (1995) 285–319
[23] N.P. Landsman and U.A. Wiedemann, Massless particles, electromagnetism, and Rieffel induction, Rev. Math. Phys. 7 n.6 (1995) 923–958
[24] P. Libermann, Problèmes d'équivalence et géométrie symplectique, Asterisque 107–108 (1983) 43–68
[25] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987)
[26] G.W. Mackey, Induced representations of locally compact groups I, Ann. Math. 55 (1952), 101–139
[27] G.W. Mackey, *The Theory of Unitary Group Representations* (University of Chicago Press, Chicago, 1976)
[28] J. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry* (Springer, New York, 1994)
[29] R. Montgomery, Canonical formulation of a classical particle in a Yang-Mills field and Wong's equations, Lett. Math. Phys. 8 (1984) 59–67
[30] G.K. Pedersen, *C^*-algebras and Their Automorphism Groups* (Academic Press, New York, 1979)
[31] M.A. Rieffel, On the uniqueness of the Heisenberg commutation relations, Duke Math. J. 39 n.4 (1972) 745–752
[32] M.A. Rieffel, Induced representations of C^*-algebras, Adv. Math. 13 (1974) 176– 257
[33] M.A. Rieffel, Deformation quantization of Heisenberg manifolds, Comm. Math. Phys. 122 (1989) 531–562
[34] M.A. Rieffel, Lie group convolution algebras as deformation quantizations of linear Poisson structures, Am. J. Math. 112 (1990) 657–686
[35] S. Sternberg, Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field, Proc. Nat. Acad. Sci. USA 74 (1977) 5253–5254
[36] W.F. Stinespring, Positive functions on C^*-algebras, Proc. Amer. Math. Soc. 6 (1955), 211–216
[37] A. Weinstein, A universal phase space for particles in Yang-Mills fields, Lett. Math. Phys. 2 (1978) 417–420
[38] A. Weinstein, The local structure of Poisson manifolds, J. Diff. Geom. 18 (1983) 523–557
[39] A. Weinstein, Affine Poisson structures, Int. J. Math. 1 (1990) 343–360
[40] P. Xu, Morita equivalent symplectic groupoids, in: *Symplectic Geometry, Groupoids, and Integrable Systems*, P. Dazord and A. Weinstein, eds. (Springer, New York, 1991) pp. 291–311
[41] P. Xu, Morita equivalence of Poisson manifolds, Comm. Math. Phys. 142 (1991) 493– 509