ON THE MASTER FIELD IN TWO DIMENSIONS

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ABSTRACT. A master field is a limiting object in the $1/N$ expansion of quantum chromodynamics. I give a general introduction to the terminology of quantum field theory. After explaining why the $1/N$ expansion is reasonable, I describe Singer's [Sin95] construction of the master field in 2 dimensions. It is followed by a more explicit construction of Gopakumar and Gross [GG95] (see also [Xu97]). In particular, I explain the relation with the free probability theory, more precisely with free Brownian motion.

I gave this talk in Marc Rieffel's seminar "Quantum Geometry" in the Fall of 1997. The main motivation for it was to understand the paper [Sin95] by Isadore Singer, "On The Master Field in Two Dimensions"; however, the results in the last section of these notes are a lot more explicit than Singer's construction. I try to give as much background as I can, but basic knowledge of operator algebras, differential geometry, stochastic processes, quantum mechanics, etc., is certainly helpful.

1. Field Theory

1.1. **Notation.** Let M be a Riemannian manifold, usually thought of as the space-imaginary time manifold.

1.1.1. *Remark.* The usual space-time Minkowski space is a pseudo-Riemannian manifold. However, if we think of time as being (purely) imaginary, the manifold becomes Riemannian, i.e. the metric becomes positive-definite. The usual justification for this action is as follows. After solving the model, we get a result which is an analytic function of time (at least for the time in the open upper-half-plane). Thus by analytic continuation it can be extended from the imaginary axis to the real axis.

1.1.2. *Notation: continued.* A *field* is a function $\phi : M \to \mathbb{C}$. A *Lagrangian* (density) is a function from fields on M to fields on M, written as $\mathcal{L}(\phi, \partial\phi)$. Usually it is the difference of the kinetic and potential energies, given by $\|\nabla\phi\|^2 - V(\phi)$, where ∇ denotes the gradient. The *action* of the Lagrangian is $S(\phi) = \int \mathcal{L}(\phi, \partial\phi)\, d\sigma$, where $d\sigma$ is the volume form. The Euler-Lagrange equations correspond to the critical points of the action, and are given by

$$\sum \partial_i \frac{\delta \mathcal{L}}{\delta(\partial_i \phi)} = \frac{\delta \mathcal{L}}{\delta \phi} \tag{1}$$

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For the (Kinetic - Potential) situation these are just $\Delta\phi = -V'(\phi)$, where Δ is the Laplacian. In particular, for the quadratic potential $V = \phi^2$ we get the equations for the harmonic oscillator.

Now we consider deformations of this structure in two directions. On one hand, we bring in a nontrivial symmetry group. On the other hand, we quantize the theory.

1.2. **Classical Yang-Mills theory.** [May77, Dre77, FS91] Let G be a (perhaps compact, not necessarily semisimple) Lie group; the physically relevant situation is $G = SU(3)$. Let ρ be a (finite-dimensional) unitary representation of G, $G \xrightarrow{\rho} U(\mathcal{H})$. A field is now a function $\phi : M \to \mathcal{H}$ or, more precisely, a section of the (trivial) bundle $M \times \mathcal{H} \to M$.

More generally, let P be the principal G-bundle over M and $P \times_G \mathcal{H}$ be the associated vector bundle, $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ and $\begin{array}{c} P \times_G \mathcal{H} \\ \downarrow \\ M \end{array}$, then a field ϕ is a section of the bundle $P \times_G \mathcal{H}$.

In analogy to the flat case, we would like to define a Lagrangian. However, for a section ϕ the quantity $\partial_i \phi$ is not well defined if we have no local horizontal direction. Thus we need to make a choice of a connection. Let A be a connection 1-form, i.e. a $\mathfrak{g}$-valued 1-form on M, where $\mathfrak{g}$ is the Lie algebra of G. The horizontal subspace is then given by the null space of this form. Denoting by ∇_A the covariant derivative with respect to A, the quantity $\|\nabla_A \phi\|^2 - V(\phi)$ is now well-defined. However, this is not quite what we want. For the moment, let's go back to the situation of a trivial bundle $M \times G$. Even though the bundle is trivial, one can choose the zero section in it in different ways. However, this choice does not affect the geometry and so should not affect the physics, i.e. the Lagrangian, either. The change of the zero section from the standard one to A corresponds to the addition to the Lagrangian of the term $\|\nabla_A A\|^2$. By definition, the quantity $F_A = \nabla_A A$ is the curvature of the connection A, a $\mathfrak{g}$-valued 2-form on M. Thus a Lagrangian which generalizes the usual one and which does not depend on the choice of the zero section, i.e. is gauge-invariant, is

$$\|\nabla_A \phi\|^2 - V(\phi) + \|F_A\|^2 \tag{2}$$

1.2.1. *Remark.* The norm of a 2-form appearing above can be described as follows. Abstractly, we have the Hodge star operator from k-forms to $(n-k)$-forms, where $n = \dim M$, $* : \Omega^k(M) \to \Omega^{n-k}(M)$. Then $\|F\|^2 = *(F \wedge *F)$.

More concretely [Sen92], let π denote the projection map $P \xrightarrow{\pi} M$, as well as its derivative. Let $\{e_i\}$ be a collection of vectors in $T_p P$, for some $p \in \pi^{-1}(x)$ such that the vectors $d\pi(e_1), \ldots, d\pi(e_n)$ form an orthonormal basis in $T_x M$. Then the following quantity is independent of the choice of e_i-s:

$$\|F\|^2(x) = \sum_{i<j} \|F(e_i, e_j)\|^2_{\mathfrak{g}}$$

where $\|\cdot\|_{\mathfrak{g}}$ is the norm on the Lie algebra coming from the inner product given by the Killing form (which should be used in the formula in the previous paragraph as well). In the case of $SU(3)$, the Lie algebra $\mathfrak{su}(3)$ is the algebra of 3×3 antisymmetric matrices, and the Killing form is the usual $-\operatorname{Tr}(T^*S)$.

1.2.2. *Terminology.* A short dictionary:

$\phi =$ matter field,	electrons	quarks
$A =$ gauge field,	photons	gluons
$F =$ field strength	electromagnetic force	strong force
theory	EM (electromagnetism)	QCD (quantum chromodynamics)
group	$U(1)$	SU(3)

The gauge field is said to mediate the interaction of the matter field.

1.2.3. *Yang-Mills fields: continued.* With the Yang-Mills Lagrangian (2), we can write the Euler-Lagrange equations and solve for both the matter field and the gauge field at once. In the sequel, we will consider the gauge field in the vacuum, i.e. set the matter field to be 0, and solve for the gauge field. Moreover, since in that case the Lagrangian $\|F_A\|^2$ involves just the curvature of the connection, the best we can hope for is to find the *Yang-Mills connection* up to gauge invariance (up to the choice of the zero section). Thus the actual space we will be working on is $\mathcal{A}/\mathcal{G}$, the space of connections modulo the gauge transformations.

1.3. **Quantization of a flat theory.** The quantized system should be described by the time-dependent Schrödinger equation $i\hbar\frac{\partial}{\partial t}\psi = H\psi$, where $\psi \in L^2(M)$ is the wave function, and the Hamiltonian H is usually of the form $H = -\frac{\hbar^2}{2m}\nabla^2 + V$. Remembering that we are considering our time to be imaginary (and the following construction is the second reason for doing so), the equation is $\frac{\partial\psi}{\partial t} = -H\psi$, with $H = \frac{1}{2}\Delta + V$ (up to normalization). Under appropriate conditions on H, that is, on V, the solution to the equation is simply $\psi = e^{-Ht}\psi_0$.

1.3.1. *The path integral.* First let us assume that $V = 0$. Also, to simplify notation consider $M = \mathbb{R}$; however, the results hold in full generality. Then $e^{-\frac{\Delta t}{2}}$ is the heat semigroup on $\mathbb{R}$, i.e. this is the operator of convolution with the gaussian function $\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$,

$$\left(e^{-\frac{\Delta t}{2}} f\right)(y) = \int f(x)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\,dx \tag{3}$$

It is well known (see [RS75] or [Str93] or [Gol76]) that the above convolution operator can also be written as the following integral over the space $\Gamma(y)$ of all continuous paths starting at y:

$$\left(e^{-\frac{\Delta t}{2}} f\right)(y) = \int_{\gamma\in\Gamma(y)} f(\gamma(t))\,dW(\gamma) \tag{4}$$

Here dW is the *Wiener measure* on the space of paths. The heuristic expression for it is $dW(\gamma) = e^{-\int\dot\gamma^2/2}\,D\gamma$, that is, if there did exist a Lebesgue measure $D\gamma$

on the space $C(\mathbb{R})$ (which of course there does not), then the Radon-Nikodym derivative of the Wiener measure with respect to it would be $e^{-\int \dot{\gamma}^2/2}$. The justification for this can be seen as follows. Think of an $(n+1)$-tuple of points $\{x_0 = y, x_1, x_2, \ldots, x_n\}$ as a discretization of a path in $\mathbb{R}$ starting at y. It is not hard to see that

$$\left(e^{-\frac{\Delta t}{2}} f\right)(y) = \frac{\sqrt{n}}{\sqrt{2\pi t}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_n) e^{-\frac{\sum |x_i - x_{i-1}|^2}{2(t/n)}}\, dx_1\, dx_2 \cdots dx_n$$

Now "take the limit". One can prove the existence of the Wiener measure in this way, although the usual ways (using martingale convergence theorems) are easier.

Finally, by using Trotter product formula we get the *Feynman-Kac* formula, which tells us that even when $V \neq 0$, under appropriate conditions on it we have a meaningful expression

$$\left(e^{-\frac{\Delta t}{2} - V} f\right)(y) = \int_{\Gamma(y)} f(\gamma(t)) e^{-\int_0^t V(\gamma(s))\, ds}\, dW(\gamma) \tag{5}$$

1.4. **Quantization of the Yang-Mills theory.** We follow [Sin95]. Although it is not obvious, $\mathcal{A}/\mathcal{G}$ is actually an (infinite-dimensional) Riemannian manifold. Thus a Laplacian is defined on it, and we can consider the Lagrangian $L(A) = \frac{\Delta}{2} A + \|F_A\|^2$, that is, $V(A) = \|F_A\|^2$. Since the manifold is not finite-dimensional the path integral formalism cannot be applied rigorously, but we can write down the heuristic expression

$$\left(e^{-\frac{\Delta t}{2} - \|F_\cdot\|^2} f\right)(A) = \int_{\Gamma(A)} f(\gamma(t)) e^{-\int_0^t \|F_{\gamma(s)}\|^2\, ds} e^{-\int_0^t \dot{\gamma}^2(s)\, ds}\, D\gamma \tag{6}$$

Now in the Yang-Mills theory we were considering the spacetime manifold, while we were quantizing the space manifold. In order to put the time into the manifold, we do the following [Sin95]. A path γ of connections on M is a connection on $M \times [0,T]$. Denoting this connection ω, its curvature is precisely $\mathcal{F}_\omega = dt \wedge \dot{\gamma} + F_{\gamma(t)}$, and so $\|\mathcal{F}_\omega\|^2 = \dot{\gamma}^2 + \int_0^t \|F_{\gamma(s)}\|^2\, ds$. Then the right-hand-side of the equation (6) becomes simply

$$\int D\omega e^{-\|\mathcal{F}_\omega\|^2/2} \tag{7}$$

This is the main object we will be working with; of course, one of the main questions is whether it is well-defined.

1.4.1. *Remark.* One usually thinks of quantum mechanics as a theory of operators of Hilbert spaces or, in the C^*-algebraic approach, as operators without Hilbert spaces. Notice that in the approach above, there are no operators present, and the evolution of the system is described by a measure on the space of connections, that is, by a stochastic connection. Euler-Lagrange equations (1) become a stochastic differential equation. This field is often called stochastic geometry (see [Sen92] and references therein). Other names associated with it are Jürg Fröhlich, Bruce Driver, and Dana Fine.

1.5. **The setup.** We are given a Riemannian manifold M and the principal bundle P over it with the internal symmetry group $G = SU(3)$, a representation ρ of the group, usually taken to be the standard representation on $\mathbb{C}^3$, and a character of that representation $\chi_\rho : G \to \mathbb{C}$, usually taken to be the trace. On the space $\mathcal{A}/\mathcal{G}$ of connections on P modulo gauge transformations, we want to make sense out of a measure given by the heuristic expression $DAe^{-\frac{1}{4g^2}||F_A||^2}$, where g is some coupling constant. Now, when one works with measures, it is often convenient to consider them as Borel measures, i.e. functionals on the space of continuous functions. Thus let C be a piecewise smooth loop in M. Define $W_C = \chi(P_{C,A})$, where $P_{C,A}$ is the holonomy of the connection A around the loop C. Note that the collection of all W_C, indexed by all loops and considered as functions on $\mathcal{A}$, separates points. Define the *Wilson loop functional* to be

$$\langle W_C \rangle = \frac{1}{Z}\int_{\mathcal{A}} DAe^{-\frac{1}{4g^2}||F_A||^2} W_C(A) \tag{8}$$

where Z is the normalization constant (hopefully finite). We can expect that these determine the measure completely, indeed by a theorem they do, assuming the measure is well-defined (see [Sen92]).

2. The $1/N$ Expansion

There is little hope for solving the above model exactly. The usual physics approach is to solve exactly a simpler model, and then expand around that theory in a certain small parameter. One can try an expansion in the coupling constant g. However, it turns out that this constant can be renormalized out of the theory, and so the expansion in it does not make sense. Alternatively, the idea of 't Hooft going back to the beginning of QCD (quantum chromodynamics) is to take the limit as $3 \to \infty$. That is, instead of $SU(3)$ we consider general $SU(N)$, and take the limit as $N \to \infty$. There are two issues here. First, does the expansion in $1/N$ make sense? While this expansion has not been very successful so far, Witten [Wit80] gives the following analogy. In the theory of critical phenomena, one would really want to describe the theory of 3 dimensions. "(Wilson and Fisher) showed that critical phenomena are simple in four dimensions and that in $4-\epsilon$ dimensions critical phenomena can be understood by perturbation theory in ϵ. Even at $\epsilon = 1$—the original three dimensional problem—this perturbation theory is quite successful."

The more important issue is that the $1/N$ expansion is useful only if we expect the theory to somehow simplify in the limit. In order to see that this is in fact the case, we need to introduce more machinery.

2.1. **Perturbation theory.** For the moment let's go back to the usual quantum mechanics. The base manifold is $\mathbb{R}^n$; the Hamiltonian is $H = \Delta + V(x)$. We consider the potential $V(x)$ as an infinite series in the powers of x. By changing coordinates one can always get rid of the constant and linear terms and so assume that $V(x) = x^2 + o(x^2)$. Thus let us write $H = (\Delta + x^2) + V(x)$, with $V(x) = o(x^2)$.

The reason for this is that for $V = 0$, this is the harmonic oscillator Hamiltonian, and the model can be solved exactly.

2.2. **The symmetric Fock space.** We follow [RS75]. First let's consider the construction in coordinates. Fix k objects a_i (annihilation operators) and their formal adjoints a_i^* (creation operators). Require them to satisfy the commutation relations

$$[a_i, a_j] = 0 = [a_i^*, a_j^*] \tag{9}$$

$$[a_i, a_j^*] = \delta(i,j)1 \tag{10}$$

where $[\cdot,\cdot]$ denotes the commutator bracket. Fix the vacuum vector ξ ($= |0\rangle$ in the physics notation). The symmetric Fock space is the closed linear span of the vectors of the form $a_{i_1}^* \cdots a_{i_n}^* \xi$. The operators act on these words according to the commutation relations and the extra condition $a_i \xi = 0$.

One can write the harmonic oscillator Hamiltonian as $H = \sum a_i a_i^* - \frac{k}{2}$ (the constant is often renormalized away), in which case the above basis for the Fock space is precisely all the eigenvectors of the Hamiltonian, with known eigenvalues.

2.2.1. *The second quantization functor.* As an aside, we give a more functorial construction of the Fock space. Let $\mathcal{H}$ be a Hilbert space. Define the full Fock space to be the sum of tensor powers of $\mathcal{H}$,

$$\mathcal{F}(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} \tag{11}$$

where we take the closure of the infinite direct sum of Hilbert spaces in the usual way. The vacuum vector ξ is the generator of the 0-th order component. For any $h \in \mathcal{H}$, define two operators on $\mathcal{F}(\mathcal{H})$ by

$$b^*(h) : \xi_1 \otimes \cdots \otimes \xi_n \mapsto h \otimes \xi_1 \otimes \cdots \otimes \xi_n \tag{12}$$

$$b(h) : \xi_1 \otimes \cdots \otimes \xi_n \mapsto \langle h, \xi_1 \rangle \xi_2 \otimes \cdots \otimes \xi_n \tag{13}$$

with the obvious modification for the 0-th and 1-st components.

Now let S_n be the symmetrization operator on $\mathcal{H}^{\otimes n}$, i.e. the projection onto the symmetric part. Then $S := \bigoplus_{n=0}^{\infty} S_n$ is the projection of $\mathcal{F}(\mathcal{H})$ onto the *symmetric Fock space* $\mathcal{F}_s(\mathcal{H})$. Also, denote by F_0 the span of finite symmetric tensor powers of $\mathcal{H}$, with no closure; this is the finite particle subspace. On $\mathcal{F}_s(\mathcal{H})$, define the *number*, *annihilation*, *creation* and *Segal field* operators [RS75] by, respectively,

$$N : \xi_1 \otimes \xi_2 \otimes \cdots \xi_n \mapsto n \tag{14}$$

$$a(h) = \sqrt{N+1}\, b(h) \tag{15}$$

$$a^*(h) = S b^*(h) \sqrt{N+1} \tag{16}$$

$$s(h) = \frac{1}{\sqrt{2}}(a(h) + a^*(h)) \tag{17}$$

All four operators are defined on F_0 and are unbounded, densely defined, closable operators on $\mathcal{F}_s(\mathcal{H})$. The map $h \mapsto s(h)$ is the second quantization functor. It is actually a functor in the sense that one also has the map of morphisms, and that is usually the most important part, but it does not concern us here. We get the construction in the previous subsection by choosing an orthonormal basis for the Hilbert space. For an explicit construction of the symmetric Fock space as well as for an explanation why this functor is properly called the *gaussian functor*, see e.g. [Fol89], with some comments in [RS75, VDN92].

2.2.2. *Perturbation theory: continued.* One makes the following completeness assumption (known in physics as an axiom): the collection of the eigenvalues of the harmonic oscillator Hamiltonian is in fact the basis for the whole Hilbert space. That is, the Hilbert space on which the perturbed Hamiltonian acts is in fact the Fock space of the unperturbed Hamiltonian. Since the position operator is, up to a constant, the real part of the creation operator, the full Hamiltonian is a certain infinite series in the various creation and annihilation operators. Thus in order to describe the evolution of the system, we have to learn to calculate quantities of the form $\langle a^{\cdot}_{i_1} a^{\cdot}_{i_2} \cdots a^{\cdot}_{i_n} \xi, \xi\rangle$, where $\cdot$ can stand for $*$ or nothing. From the defining relations it is clear that such an expression is not zero only if all the operators in the word can be paired off so that each pair consists of an annihilation operator followed by the creation operator with the same index. See Figure 1 for a pictorial example.

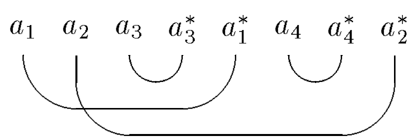

FIGURE 1. A sample pair partition of eight letters

2.3. **Matrices; Diagrams.** In gauge theory, our Hamiltonian is matrix-valued; therefore when quantized, it becomes a matrix of creation and annihilation operators. Consider as an example the expression $\operatorname{tr}(M^8)$. Here M is an $N \times N$ matrix, $\operatorname{tr} = \frac{1}{N}\operatorname{Tr}\langle \cdot \xi, \xi\rangle$ is the normalized trace, the matrix is hermitian, and except for this symmetry its entries are creation and annihilation operators with different indices. Then

$$\operatorname{tr}(M^8) = \frac{1}{N} \sum_{i_1, i_2, \ldots, i_8} \langle M_{i_1,i_2} M_{i_2,i_3} \cdots M_{i_8,i_1} \xi, \xi \rangle \tag{18}$$

with the sum taken over all N^N possible combinations of indices. Again, only those terms in the sum will be nonzero which can be divided into annihilation-creation pairs as above. In this case, $M_{i,j} = M^*_{k,l}$ iff $i = l, j = k$. That is, we connect $M_{i,j}$ with $M_{k,l}$ by a double line, corresponding to a pair of indices. See Figure 2 for an example. Then the quantity $\operatorname{tr}(M^8)$ is the sum over all such partitions,

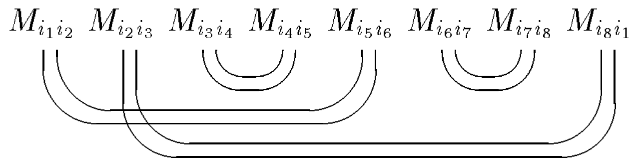

FIGURE 2

with weights corresponding to how often each partition occurs, i.e. for how many possible collections of indices it occurs.

There are various graphical methods for representing such partitions. One can leave them as partitions (Figures 1,2). One can glue the set on which the partition is taken into a circle, with the lines inside (Figure 3). Thinking of the partition as identifying the edges of an n-gon so that the identified edges have opposite orientations (Figure 4), one gets a graph corresponding to the partition, as in Figure 5. This is the method used by Voiculescu [VDN92]; there is a very clear exposition of it in [Shl97]. We mostly follow [BIZ80], who basically perform an inversion with respect to the origin in the above picture. That is, they contract the set which is being partitioned into a single vertex and consider the partition lines as emanating from that vertex. See Figures 6,7 for examples.

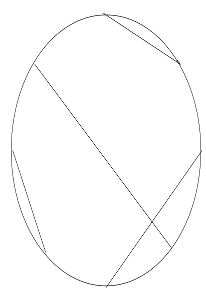

FIGURE 3. The partition of eight points on a circle.

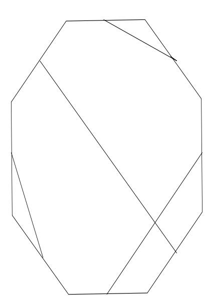

FIGURE 4. The partition of the edges of an octagon.

Let us now see how many collections of indices correspond to such a diagram. There is precisely one free parameter corresponding to each loop in the diagram

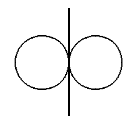

FIGURE 5. The graph obtained by identifying the edges.

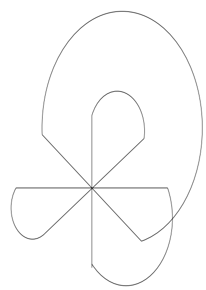

FIGURE 6. The diagram with single edges.

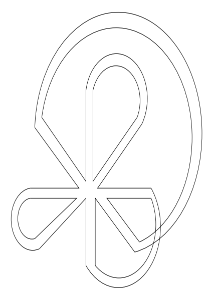

FIGURE 7. The diagram with double edges.

(once one index of an entry on the loop is determined, all the rest have to equal to it), which gives us N possible values for the index per loop (physicists say that there are N gluons running around the loop). Also, we are considering the normalized trace, so there is an overall factor of $1/N$. In our example, there are three loops, and so the power is $N^3\frac{1}{N} = N^2$.

In general to an expression of the form

$$\langle \operatorname{tr}(M^{n_1}) \operatorname{tr}(M^{n_2}) \cdots \operatorname{tr}(M^{n_k}) \rangle \tag{19}$$

there will correspond the diagrams of the following shape. They will have k vertices, with a weight factor of $1/N$ for each vertex. i-th vertex will have n_i edges emanating out of it. Finally, it will have a certain number of loops, with a factor of N corresponding to each loop. Denote by V, E, F the number of vertices, edges, and faces, respectively, of the diagram. Denote by χ its Euler characteristic, by C the number of the connected components, and by n the total degree $n = n_1 + n_2 + \cdots + n_k$. Then the weight of the diagram is given by N^{F-V}.

But

$$F - V = C \cdot \chi - 2V + E = C \cdot \chi - 2V + n/2 \tag{20}$$

Both n and V are determined by the moment (19) we are calculating, while C and χ vary from diagram to diagram.

Now remember that we are interested in the limit $N \to \infty$. In this limit, only those diagrams contribute to the calculation which have the highest power of N as the weight. Now the largest Euler characteristic is the one of the sphere (the plane). That is, in the limit the diagrams that "survive" are precisely the planar diagrams (those that can be drawn in the plain with no crossings; they are often called noncrossing diagrams).

But in fact more is true. Suppose we are given two quantities X, Y, each of which is a moment like (19). Let us look at their covariance, $\langle XY \rangle - \langle X \rangle \langle Y \rangle$. The second quantity is the sum over all pairs of diagrams, one coming from the expansion of X and the other coming from the expansion of Y, with weights being products of weight of the corresponding diagrams. That is, it is a sum over diagrams with at least two components. Normally, $\langle XY \rangle$ is different since its expansion involves connected diagrams as well. However, remember that in the limit, only diagrams with the largest $C\chi$ enter. That is, in the limit only the maximally disconnected diagrams enter in the expansion. This is true for both quantities, and so in the limit $\langle XY \rangle = \langle X \rangle \langle Y \rangle$.

In probability theory this would mean that the quantities X, Y are (asymptotically) uncorrelated. Here the statement means that the functional $\langle \cdot \rangle$ is a multiplicative functional. Now a multiplicative functional is a point in the underlying space. Remembering that our functional is the measure on the space of connections, we would thus expect that in the limit it is given by a single connection. That is, in the large N limit the quantum Yang-Mills theory becomes a classical Yang-Mills theory. This is the desired simplification.

3. The Master Field

The previous section is supposed to indicate that we can expect the existence of a *master field*, i.e. a group G_∞, a principal G_∞-bundle P_∞ over M, with a (single) connection A_∞ on it, and a character tr_∞ of some representation of G_∞ so that

$$\lim_{N \to \infty} \langle W_C \rangle = \mathrm{tr}_\infty P_{C, A_\infty} \tag{21}$$

3.1. **Singer's construction.**

3.1.1. *Preliminaries.* We follow [Sin95]. We can consider $\langle W_C \rangle$ as a functional on the group of loops. Taking the group of loops as our G_∞ and this functional as tr_∞, we get the master field. The following is this idea made precise.

Fix $x \in M$. Let $\mathcal{F}_x$ be the space of piecewise smooth paths in M starting at x. Let Ω_x be the subset of loops in $\mathcal{F}_x$, i.e. the paths ending at x. Ω_x is a group under concatenation. The next question is what kind of equivalence do we put on the space of paths. If we identify homotopic paths we get the fundamental group of M. It turns out that this is too small. We'll see that the important functionals will depend on the area bound by a loop but not on the loop itself. Therefore we put on $\mathcal{F}_x$ the backtracking equivalence (the notion is due to Kobayashi). That is, we identify two paths which differ by a (finite number of) smooth backtracking segments along the trajectory of the path. See [Sin95] and references therein. Let $\tilde{\mathcal{F}}_x$, $\tilde{\Omega}_x$ be the corresponding quotients. We'll call $\tilde{\Omega}_x$ the loop group of M; note that this is not the usual notion of a loop group. It is easy to see that the Wilson loop functional $\langle W_C \rangle$ projects to $\tilde{\Omega}_x$.

$\tilde{\mathcal{F}}_x$ is the principal $\tilde{\Omega}_x$-bundle over M. It has a tautological connection, which is defined by the following instructions for horizontal lifting: for a path γ in M, its horizontal lift starting at a loop $C \in \tilde{\Omega}_x$ has, in the fiber over $\gamma(t)$, the value $C + \gamma|_{[0,t]}$.

Given a principal G-bundle $P \to M$ with a connection A, the map $\gamma \mapsto P_{\gamma,A}$ induces a homomorphism $\tilde{\Omega}_x \to G$. Conversely, we have the following Kobayashi construction: given a homomorphism $\phi : \tilde{\Omega}_x \to G$, we get the principal G-bundle $P = \tilde{\mathcal{F}}_x \times_\phi G$, with the connection induced by the one on $\tilde{\mathcal{F}}_x$.

3.1.2. *The construction.* First and foremost, from now on we assume that M **is two dimensional**, either $\mathbb{R}^2$ or a compact surface.

We make two claims; the justifications are put off till the next section. First, we assume that the measure on $\mathcal{A}/\mathcal{G}$ is an actual measure. Second, denoting $l_N(C) = \langle W_C \rangle_N$, we assume that the limiting functional on the loop group, $l_\infty = \lim l_N$, actually exists.

The first assumption implies that each l_N is an integral with respect to a positive measure, and so is a positive functional. Therefore so is l_∞. Moreover, it is easy to see that l_∞ is central, i.e. constant on conjugacy classes. Therefore one can apply the GNS construction to the pair $(\tilde{\Omega}_x, l_\infty)$. We get a pair of an algebra (with a representation) and a trace vector, (W, e). Let $G_\infty = U(W)$, the unitary group of the algebra. Let $\mathrm{tr}_\infty = \langle \cdot e, e \rangle$, the functional induced by the trace vector. By the Kobayashi construction, the homomorphism $U_\infty : \tilde{\Omega}_x \to G_\infty$ produces a principal bundle P_∞ with a connection A_∞, so that

$$\mathrm{tr}_\infty P_{C,A_\infty} = \langle U_\infty(C) e, e \rangle = l_\infty(C) \tag{22}$$

That is, A_∞ is the master field.

3.1.3. *Question.* Singer claims that for $M = \mathbb{R}^2$, the loop group is the free group on uncountably many generators. Is that true?

3.2. **Two-dimensional Yang-Mills.** Here is the reason why we have difficulty interpreting the expression $DAe^{-\|F_A\|^2}$. The curvature F_A is a quadratic expression in the connection coefficients. Thus the above measure is of the form $dx\, e^{-x^4}$. Now on infinite-dimensional spaces, there is essentially one measure we know how to deal with, and that is the Wiener measure, with the heuristic expression $dx\, e^{-x^2}$. I do not know of an actual theorem to this effect, although the Central Limit Theorem is certainly a factor here.

3.2.1. *The trivialization.* Throughout this subsection, M is the plane $\mathbb{R}^2$. In this case, the above difficulty is circumvented, as follows. Remember that we are interested in the measure on $\mathcal{A}/\mathcal{G}$, that is, we identify connections which differ only by the choice of the zero section, and this takes care of one degree of freedom.

Any bundle over $\mathbb{R}^2$ is trivial, in particular the bundle of $\mathfrak{g}$-valued 2-forms is. Explicitly, we identify a 2-form F with a function F such that $F = F\, d(\text{volume form})$ (this is, of course, possible only in two dimensions). In order to identify the curvature form with an element of $\mathcal{A}/\mathcal{G}$, we have to fix a gauge, i.e. a zero section for each class. The usual way is to fix the radial gauge [Sen92], that is, require that the zero section consist of the endpoints of the horizontal lifts of radial segments in M.

Alternatively and a bit more explicitly, we can fix the axial gauge [GKS89]. That is, for a connection 1-form $A = A_1\, dx + A_2\, dy$, we require $A_2 = 0$. In that case $F = \partial_y A_1$, and so manifestly is linear in A.

In any case, we now have to make sense out of the measure on the space $C(\mathbb{R}^2, \mathfrak{su}(N))$ given by the expression $dFe^{-\operatorname{Tr}\|F\|_2^2}$. This can be done.

3.2.2. *Brownian sheet.* There are at least three gaussian processes indexed by $\mathbb{R}^2$ whose correlation functions generalize the correlation function of the Brownian motion in a sensible way. Here we need the one known as the *Brownian sheet*, which is defined as follows. We follow [GKS89].

For any Borel subset B of $\mathbb{R}^2$ of finite Lebesgue measure (which we will denote by $|B|$), we define an $N \times N$ matrix of random variables, $B \mapsto F(B) = [F_{ij}(B)]$. The matrix is skew-symmetric (that is, lies in $\mathfrak{su}(N)$); except for this symmetry the entries are independent gaussian variables with mean 0 and variance $|B|$. We also require that whenever $B \cap C = \emptyset$, $F(B)$ and $F(C)$ are independent. It follows that, more generally, denoting by E the expectation, we have

$$\mathrm{E}[F_{ij}(B), F_{kl}(C)] = |B \cap C| \tag{23}$$

that is, the correlation function is given by the area. Since the variables are gaussian, specifying the correlation function specifies the joint distribution of the variables.

Heuristically, we can think of F as given by $F(B) = \int_B F(x)\, dx$, except that the pointwise random variables $F(x)$ do not really exist. Nevertheless, this expression, and the Yang-Mills measure can be given meaning. Indeed, we proceed as in the case of the usual Brownian motion. Let the sample space be $\Omega = C(\mathbb{R}^2, \mathfrak{su}(N))$, and define the random variables (= measurable functions on Ω) by $F(B)(f) = \int_B f(x)\, dx$. Now define a probability measure dP on Ω by requiring that under the expectation given by this measure the random variables $F(B)$ have the correlation function (23) and be gaussian. One checks that this can be done by the Kolmogorov consistency theorem. This measure dP is precisely the Yang-Mills measure whose definition we wanted to formalize.

3.2.3. *Compact surfaces.* It is Sengupta's Ph.D. thesis [Sen92] that one can also define the Yang-Mills measure over the compact surfaces. There the bundle is no longer trivial. Very briefly, one does the following. Cut the surface into two flat pieces. On each of them, construct the measure as above. Glue the pieces back together; this means that we identify the (stochastic) holonomies around the boundaries of the two pieces. Now prove that the resulting measure is invariant under measure-preserving diffeomorphisms. This means that the measure does not depend on the initial cutting, and is in a sense canonical.

3.2.4. *Question.* One can of course, in a manner almost identical to the Brownian sheet, construct the measure on the space of functions over, say, the sphere S^2. Have the pure probability people considered Wiener-type measures on the space of sections of any non-trivial vector bundle?

3.3. **Free probability.** In this section we will construct explicit algebras of operators which form the master field, that is, whose correlations reproduce the limits of the correlations of the Yang-Mills field and the Wilson loop functionals as $N \to \infty$.

3.3.1. *Motivation.* We saw that whenever $B \cap C = \emptyset$, the corresponding random variables $F(B), F(C)$ are independent. That means that $\mathrm{E}[F(B)F(C)] = \mathrm{E}[F(B)]\,\mathrm{E}[F(C)]$. So why do we not have a multiplicative functional for finite N, why do we need to take the limit? The answer is that the random variables are matrix-valued, and the actual functional is $\mathrm{tr} \circ \mathrm{E}$. But tr is not multiplicative; in fact, there are no multiplicative functionals on $M_N(\mathbb{C})$ or $SU(N)$. However, in the limit the variables become not just independent but freely independent, and this condition is strong enough to make the functional multiplicative.

3.3.2. *The full Fock space.* [VDN92] Again, we start with a Hilbert space $\mathcal{H}$. $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, with the vacuum vector ξ being the generator of the 0-th component. Given $h \in \mathcal{H}$, we define the free creation and annihilation operators (with no symmetrization) by

$$l(h) : \xi_1 \otimes \cdots \otimes \xi_n \mapsto h \otimes \xi_1 \otimes \cdots \otimes \xi_n \tag{24}$$

$$l^*(h) : \xi_1 \otimes \cdots \otimes \xi_n \mapsto \langle h, \xi_1 \rangle \xi_2 \otimes \cdots \otimes \xi_n \tag{25}$$

with obvious modification if the vector lies in the 0-th or 1-st component. Here the inner product on $\mathcal{H}$ is linear in the second variable. Note also that the notation for the creation operator is the opposite of the one in the symmetric case. Finally, define $s(h) = \frac{1}{2}(l(h) + l^*(h))$. This is the free Gaussian functor of Voiculescu [VDN92].

3.3.3. *Digression: algebras.* One also has the third analog of the second quantization functor, and that is the fermionic functor, where one uses the antisymmetric Fock space (there is in fact a similar construction for any $q \in [-1, 1]$, but that would take us too far afield). For completeness, we give a table of algebras one gets in this fashion. If the state is not listed, it is the canonical trace-state.

(1) Bosonic case. $C^*(s(\mathcal{H})) = (L(\mathbb{R}^n), e^{-x^2/2})$, where $\mathbb{R}^n$ is considered as an additive group. $C^*(a(\mathcal{H})) = \mathcal{K}$, the algebra of compact operators; $W^*(a(\mathcal{H})) = \mathcal{B}$(separable Hilbert space).
(2) Free case. $C^*(s(\mathcal{H})) = C^*_r(\mathbb{F}_n)$, $W^*(s(\mathcal{H})) = L(\mathbb{F}_n)$, $C^*(a(\mathcal{H}))$ is the extension of the Cuntz algebra $\mathcal{O}_n$ by the compact operators for $n < \infty$, otherwise just $\mathcal{O}_\infty$.
(3) Fermionic case. $C^*(s(\mathcal{H}))$ is the Clifford algebra on n generators, $C^*(a(\mathcal{H})) = M_{2^n}$, the full matrix algebra. For $n = \infty$, $W^*(s(\mathcal{H})) = W^*(a(\mathcal{H})) = R$, the hyperfinite II_1 factor.

The operators $s(h)$ have the following special property. If, for $h, g \in \mathcal{H}$, we have $h \perp g$, then the corresponding operators $s(h), s(g)$ are freely independent.

3.3.4. *Definition.* [VDN92] Given an algebra with a state ϕ on it, and two subalgebras $\mathcal{A}, \mathcal{B}$. We say that the two subalgebras are freely independent with respect to ϕ if the following holds. For any collection of $a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$ with the property that $\phi(a_i) = 0 = \phi(b_i)$ for all i, we have $\phi(a_1 b_1 a_2 b_2 \cdots a_n b_n) = 0$.

As usual in probability theory, two elements are called (*)-freely independent is the (*)-algebras they generate are freely independent.

3.3.5. *Remark.* Note that stochastic independence $\phi(ab) = \phi(a)\phi(b)$ does not allow us to calculate moments of the form $\phi(abab)$ in terms of moments of a-s and b-s alone unless a-s and b-s commute. Suppose, however, that the algebras of a-s and b-s are freely independent. For any x, denote $\mathring{x} = x - \phi(x)$; then $\phi(\mathring{x}) = 0$. Then

$$\phi(a_1 b_1 \cdots a_n b_n) = \phi\left((\mathring{a}_1 + \phi(a_1))(\mathring{b}_1 + \phi(b_1)) \cdots (\mathring{a}_n + \phi(a_n))(\mathring{b}_n + \phi(b_n)) \right) \tag{26}$$

and this can be expressed as a polynomial in the moments of smaller orders. Note, however, that the polynomial is quite complicated, due to the fact that $\phi(\mathring{a}_1 \mathring{a}_2)$ need not be zero. For an example of a calculation, see Section 3.5.1.

3.3.6. *The full Fock space: continued.* [VDN92] Here's the reason for the "free" terminology. Let $\{e_i\}_{i=1}^{|\mathcal{H}|}$ be an orthonormal basis for $\mathcal{H}$. Then we have the following string of equalities.

$$\begin{aligned} W^*(s(\mathcal{H})) = W^*(\{s(e_i)\}_{i=1}^{|\mathcal{H}|}) = \mathop{*}_{i=1}^{|\mathcal{H}|} W^*(s(e_i)) = \mathop{*}_{i=1}^{|\mathcal{H}|} C(\mathbb{T}) \\ = \mathop{*}_{i=1}^{|\mathcal{H}|} L(\mathbb{Z}) = L\left(\mathop{*}_{i=1}^{|\mathcal{H}|} \mathbb{Z}\right) = L(\mathbb{F}_{|\mathcal{H}|}) \end{aligned} \tag{27}$$

Here $|\mathcal{H}| = \dim \mathcal{H}$, $*$ is the (appropriate) free product, L denotes the von Neumann algebra of a group, and $\mathbb{F}_n$ is the free group on n generators.

3.3.7. *The master field in two dimensions.* Finally, let $\mathcal{H} = L^2(\mathbb{R}^2)$. For a set B in $\mathbb{R}^2$ of finite Lebesgue measure, its indicator function $\mathbf{1}_B \in L^2(\mathbb{R}^2)$. Write $s(B) := S(\mathbf{1}_B)$. Then for $B \cap C = \emptyset$, the indicator functions are orthogonal and so the corresponding operators are freely independent (cf. the Brownian sheet). Then we have the following convergence in distribution. For any $B_1, B_2, \ldots, B_k \subset \mathbb{R}^2$, $n_1, n_2, \ldots, n_k \in \mathbb{N}$, we have the convergence of moments

$$\operatorname{tr} \mathrm{E}\,[F^{n_1}(B_1)F^{n_2}(B_2)\cdots F^{n_k}(B_k)] \stackrel{N\to\infty}{\longrightarrow} \langle s^{n_1}(B_1)s^{n_2}(B_2)\cdots s^{n_k}(B_k)\xi, \xi\rangle \tag{28}$$

Thus the pair $(L(\mathbb{F}_\infty), \langle\cdot\xi, \xi\rangle)$ deserves to be called the master field in two dimensions. Note that the infinity here is countable. For results of this type, see [VDN92, GG95, Bia97, Xu97] and references therein.

3.4. **Holonomy.** In order to see the limiting behaviour of the Wilson loop functionals and make Singer's construction more precise, we first have to look at the holonomy for finite N. Consider a loop C which bounds a region B. Then by Stokes theorem

$$\int_B F_A d\sigma = \int_C dA \tag{29}$$

where σ is the area form. Thus

$$V_C = \exp \int_B F_A d\sigma = \exp \int_C dA \tag{30}$$

However, this is not the holonomy; the holonomy is the " multiplicative integral" of $\exp(dA)$ around the loop C. That is, it is a limit of the Riemann products, defined in the same way as Riemann sums. Physicists speak of "ordered products". I am not aware of a standard mathematical notation for these, I'll denote them by

$$U_C = \left(\int \prod\right)_C \exp(dA) \tag{31}$$

The reason for the lack of classical notation is that if the A-s did commute then the multiplicative integral of the exponential would be the exponential of the additive integral. However, A-s do not commute, and contrary to a widely held belief $\log(ab) \neq \log(a) + \log(b)$.

Nevertheless, there is a relation between V_C and U_C. Namely, for small loops (i.e. loops that bound small area), they are very close to each other; that is, in a sense they have the same generator. Note also that U_C gives a representation of $\tilde{\Omega}_x$, that is, it is a "group of unitary operators indexed by $\tilde{\Omega}_x$", while V_C is not. Motivated by the Chernoff product formula [Gol76], we have the following

3.4.1. *Question.* Is there a true form of the following theorem?

Given a group G, with a norm functional $|\cdot|$ on it, and the topology induced by this functional. And given a finite von Neumann algebra (W, tr). Let U be a unitary representation of G in W, and V be a family of unitary operators indexed by G, both continuous when W is considered with topology induced by tr (or L^1 or L^2?). Suppose the two families have the same generator, that is,

$$\frac{V_g - 1}{|g|} \xrightarrow{d} A \tag{32}$$

and the same for U. Here $\xrightarrow{d}$ means convergence in distribution, that is, convergence of moments with respect to the trace. More generally, $T_n \xrightarrow{d} T$ means weak convergence of measures $\operatorname{tr}(E(T_n; \cdot))$ where $E(T; \cdot)$ is the spectral measure. Then can one conclude that

$$\lim_{\substack{|g_i| \to 0 \\ g_1 g_2 \cdots g_n = g}} V_{g_1} V_{g_2} \cdots V_{g_n} \stackrel{d}{=} U_g \tag{33}$$

3.4.2. *Free multiplicative Brownian motion.* Even if the above theorem is not true in any kind of generality, in our case it is clear that by this method the (random) holonomy around a simple loop C (i.e. a loop with no self-intersections) is in the closure of the algebra generated by the the curvature corresponding to the region bounded by C. Therefore if the regions bounded by C_1, C_2 do not overlap, the corresponding holonomy random variables are independent, and their limits are freely independent. Denote these limits again by U_C. Then we have the following information about U. First, it is a multiplicative family, $U_{C_1} U_{C_2} = U_{C_1 C_2}$. Second, whenever their regions do not overlap, the variables are freely independent. Third, the distribution of the variable depends only on the area bounded by the loop, and moreover scales as the square root of the area. These conditions suffice to identify U_C as the free multiplicative Brownian motion. Indeed, take a simple region B and divide it into n simple regions B_i of area $\frac{1}{n}$. Also let B^i be regions of the same area as B but disjoint (independent) from it and each other. Then

$$U_B = U_{B_1} \cdots U_{B_n} \stackrel{d}{=} \sqrt[\sqrt{n}]{U_{B^1}} \cdots \sqrt[\sqrt{n}]{U_{B^n}} = \sqrt[\sqrt{n}]{U_{B^1} \cdots U_{B^n}} \tag{34}$$

where $\stackrel{d}{=}$ means the equality in distribution. This kind of stability identifies the free multiplicative gaussian distribution uniquely. Note that we do not (yet) have the free multiplicative Central Limit Theorem; however, the corresponding limit distribution has been identified. Specifically, for a loop U which bounds area A, the moments are given in terms of Laguerre polynomials by

$$\operatorname{tr}(U^n) = \frac{1}{n} L^1_{n-1}(nA) e^{-n\frac{A}{2}} \tag{35}$$

For the S-transform machinery of Voiculescu that is used to compute the distribution, see [VDN92, GG95, Bia97]. For rigorous results about the 1-dimensional free multiplicative Brownian motion, see [Bia97] and references therein.

3.5. **Examples.** The fact that non-overlapping simple loops correspond to free random variables allows us to calculate moments of arbitrary loops. The following examples are stolen from [GG95].

3.5.1. *Preliminaries.* We record two moments that will be needed. First, from (26) we get with no assumptions that

$$\phi(ax) = \phi(\mathring{a}\mathring{x}) + \phi(a)\phi(x) \tag{36}$$

Now assume $\{a, b\}$ are freely independent form $\{x, y\}$. Then

$$\phi(ax) = \phi(a)\phi(x)$$

and

$$\begin{aligned}\phi(axby) &= \phi(x)\phi(\mathring{a}\mathring{x}\mathring{y}) + \phi(b)\phi(\mathring{a}\mathring{x}\mathring{y}) + \phi(a)\phi(b)\phi(\mathring{x}\mathring{y}) + \phi(\mathring{\mathring{a}b})\phi(x)\phi(y) \\ &\quad + \phi(a)\phi(x)\phi(b)\phi(y) \\ &= \phi(x)\phi(\mathring{\mathring{a}b})\phi(\mathring{y}) + \phi(b)\phi(\mathring{a})\phi(\mathring{x}\mathring{y}) + \phi(a)\phi(b)(\phi(xy) - \phi(x)\phi(y)) \\ &\quad + (\phi(ab) - \phi(a)\phi(b))\phi(x)\phi(y) + \phi(a)\phi(x)\phi(b)\phi(y) \\ &= \phi(a)\phi(b)\phi(xy) - \phi(a)\phi(b)\phi(x)\phi(y) + \phi(ab)\phi(x)\phi(y)\end{aligned}$$

3.5.2. *Remark.* Gopakumar and Gross [GG95] state that for U, V corresponding to two simple non-overlapping loops, one has

$$\begin{aligned}&\operatorname{tr}(U^{n_1}V^{m_1}U^{n_2}V^{m_2}\ldots U^{n_{k-1}}V^{m_{k-1}}U^{n_k}V^{m_k}) \\ &\qquad = \operatorname{tr}(U^{n_1})\operatorname{tr}(V^{m_1}U^{n_2}V^{m_2}\ldots U^{n_{k-1}}V^{m_{k-1}}U^{n_k}V^{m_k}) \\ &\qquad\quad - \operatorname{tr}(U^{n_1}V^{m_1})\operatorname{tr}(U^{n_2}V^{m_2}\ldots U^{n_{k-1}}V^{m_{k-1}}U^{n_k}V^{m_k}) \\ &\qquad\quad + \ldots - \operatorname{tr}(U^{n_1}V^{m_1}U^{n_2}V^{m_2}\ldots U^{n_{k-1}}V^{m_{k-1}})\operatorname{tr}(U^{n_k}V^{m_k}) \\ &\qquad\quad + \operatorname{tr}(U^{n_1}V^{m_1}U^{n_2}V^{m_2}\ldots U^{n_{k-1}}V^{m_{k-1}}U^{n_k})\operatorname{tr}(V^{m_k})\end{aligned}$$

If true, this simplifies the general calculations considerably. I do not know of a proof or a counterexample to this statement.

3.5.3. *The figure eight.* First we consider non-overlapping loops. The picture is in Figure 8.

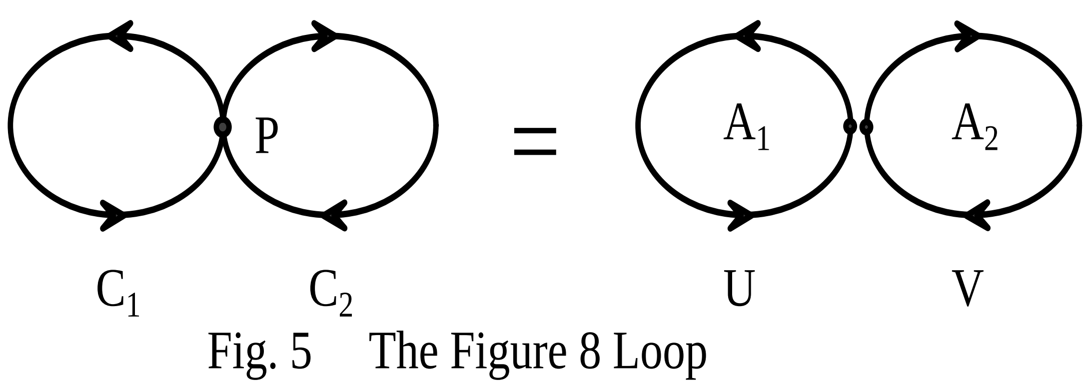

FIGURE 8

$$U_C = UV$$
$$\mathrm{tr}(U_C) = \mathrm{tr}(UV) = \mathrm{tr}(U)\,\mathrm{tr}(V) = e^{-A_1/2}e^{-A_2/2} = e^{-\frac{A_1+A_2}{2}}$$

3.5.4. *Overlapping loops.* A simple pair of overlapping loops is given in Figure 9. The loops separate two bounded regions from the plane. The operators corresponding to (the boundaries of) those regions are freely independent, so we decompose the given loop in terms of these generators.

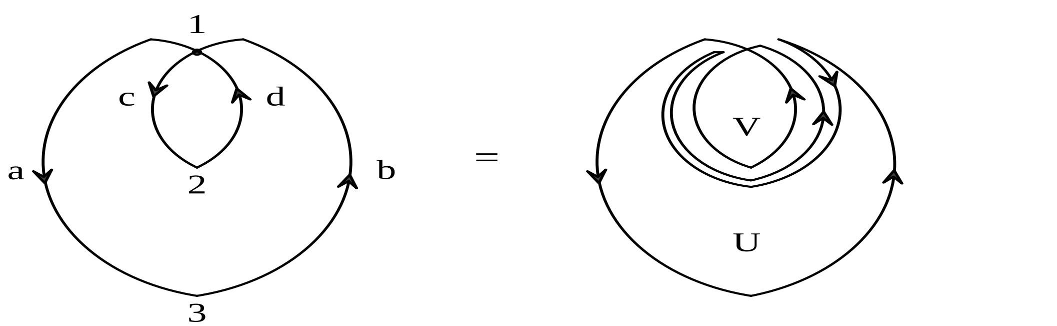

Fig. 6a Overlapping Loops Fig. 6b Simple Loop Decomposition

FIGURE 9

$$U_C = UV^2$$
$$\mathrm{tr}(U_C) = \mathrm{tr}(UV^2) = \mathrm{tr}(U)\,\mathrm{tr}(V^2) = e^{-\frac{A_1}{2}}e^{-A_2}(1 - A_2)$$

3.5.5. *Overlapping loops: 2.* A more complicated example is given in Figure 10.

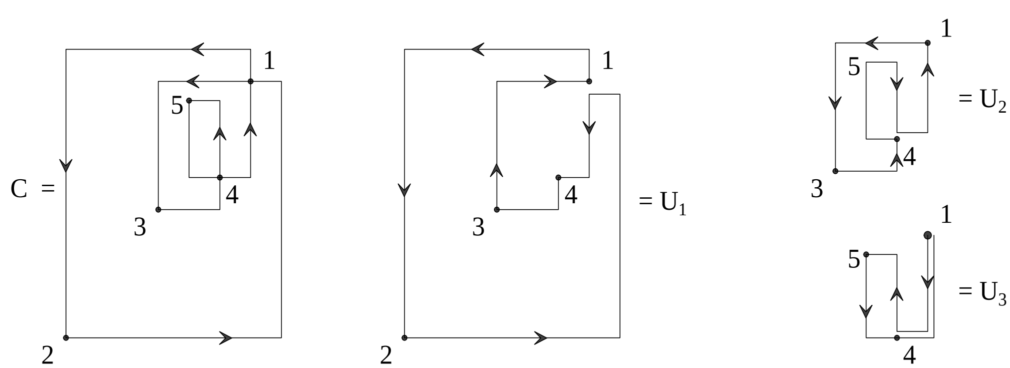

Fig. 7 A Loop and its Decomposition Into Simple Loops

FIGURE 10

$$\begin{aligned} U_C &= U_1 U_2 U_3 U_2 U_3^2 \\ \operatorname{tr}(U_C) &= \operatorname{tr}(U_1 U_2 U_3 U_2 U_3^2) = \operatorname{tr}(U_1)\operatorname{tr}(U_2 U_3 U_2 U_3^2) \\ &= \operatorname{tr}(U_1)\left(\operatorname{tr}(U_2)^2 \operatorname{tr}(U_3^3) - \operatorname{tr}(U_2)^2 \operatorname{tr}(U_3)\operatorname{tr}(U_3^2) + \operatorname{tr}(U_2^2)\operatorname{tr}(U_3)\operatorname{tr}(U_3^2)\right) \\ &= e^{-(\frac{1}{2}A_1 + A_2 + \frac{3}{2}A_3)}\left(1 - 3A_3 + \frac{3}{2}A_3^2 - (1-A_3) + (1-A_3)(1-A_2)\right) \end{aligned}$$

3.5.6. *A non-planar diagram.* Finally, a non-planar example is given in Figure 11.

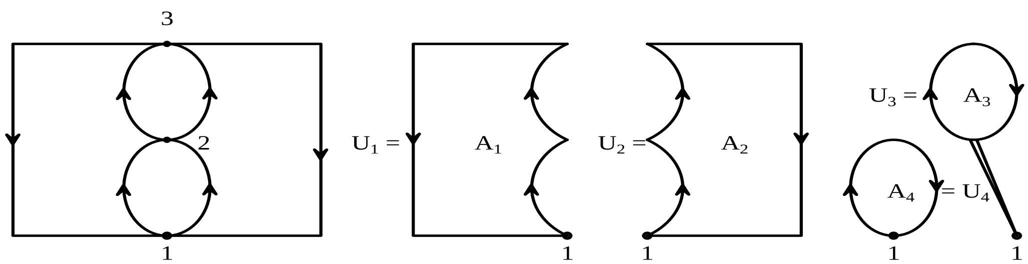

FIGURE 11. A nonplanar graph decomposed into simple loops

$$\begin{aligned} U_C &= U_3 U_2 U_4^{-1} U_3^{-1} U_4 U_1 \\ \operatorname{tr}(U_C) &= \operatorname{tr}(U_3 U_2 U_4^{-1} U_3^{-1} U_4 U_1) = \operatorname{tr}(U_3 U_2 U_4^{-1} U_3^{-1} U_4)\operatorname{tr}(U_1) \\ &= \operatorname{tr}(U_1)\operatorname{tr}(U_2)\operatorname{tr}(U_4^{-1} U_3^{-1} U_4 U_3) \\ &= \operatorname{tr}(U_1)\operatorname{tr}(U_2)\left(\operatorname{tr}(U_4^{-1})\operatorname{tr}(U_4) - \operatorname{tr}(U_4^{-1})\operatorname{tr}(U_3^{-1})\operatorname{tr}(U_4)\operatorname{tr}(U_3)\right. \\ &\qquad \left. + \operatorname{tr}(U_3^{-1}) tr(U_3)\right) \\ &= e^{-\frac{A_1+A_2}{2}}\left(e^{-A_4} - e^{-(A_4+A_3)} + e^{-A_3}\right) \end{aligned}$$

3.5.7. *Remark.* Such loop computations have also appeared in the physics literature, see [KK81] and their references. It would be interesting to see how free independence comes into their picture, but I do not understand their arguments at all.

3.5.8. *Question.* We reiterate the question 3.1.3: what is the group of unitary operators comprising the two-dimensional free multiplicative Brownian motion? In fact, what is the group of unitary operators comprising the usual free multiplicative Brownian motion? What group is generated by $\{\exp is(h) : h \in \mathcal{H}\}$?

3.5.9. *References.* There are two references for the precise results. Philippe Biane [Bia97] deals with the 1-dimensional free multiplicative Brownian motion and explains how it is generated by the free additive Brownian motion and in which sense it is the limit of matrix-valued Brownian motions. In a preprint [Xu97], about which I learned only after giving the talks, Feng Xu does the following. He gives rigorous descriptions of results in sections 2.2.2–2.3, 3.3.7–3.4.2. He also indicates the relation between holonomy and curvature alluded to in Section 3.4, namely that if the curvature is a Gaussian random variable then the distribution of the holonomy is the heat kernel measure on the unitary group.

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