

From random matrices to free groups, through non-crossing partitions

Michael Anshelevich

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RANDOM MATRICES

For each N , $A^{(N)}, B^{(N)}$ = independent $N \times N$ symmetric Gaussian random matrices, i.e. $A_{ij}^{(N)} = A_{ji}^{(N)}$, otherwise independent Gaussian $\mathcal{N}(0, \frac{1}{N})$.

Arise in: nuclear physics, quantum chaos, communication theory, string theory.

Want joint moments

$$\frac{1}{N} \text{Tr}[p_1(A^{(N)})q_1(B^{(N)}) \dots p_k(A^{(N)})q_k(B^{(N)})].$$

Example .

$$M^{(N)} = \frac{1}{N} \text{Tr}[(A^{(N)})^2 B^{(N)} A^{(N)} B^{(N)}].$$

Note: this is a random variable.

Theorem . (Wigner ~50)

$$\frac{1}{N} \text{Tr}[(A^{(N)})^k] \xrightarrow{N \rightarrow \infty} \int x^k d\sigma(x)$$

the moments of the semicircular distribution

$$d\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{[-2,2]}(x) dx.$$

Theorem . (Voiculescu '91) If $\int p_i(x) d\sigma(x) = 0$,
 $\int q_i(x) d\sigma(x) = 0$, then

$$\frac{1}{N} \text{Tr}[p_1(A^{(N)})q_1(B^{(N)}) \dots p_k(A^{(N)})q_k(B^{(N)})] \xrightarrow{N \rightarrow \infty} 0.$$

Note this is enough to find all moments, e.g.

$$\begin{aligned} M^{(N)} &\sim \frac{1}{N} \text{Tr}[(A^{(N)})^2 - 1] B^{(N)} A^{(N)} B^{(N)} \\ &\quad + \frac{1}{N} \text{Tr}[1] \frac{1}{N} \text{Tr}[B^{(N)} A^{(N)} B^{(N)}] \rightarrow 0. \end{aligned}$$

FREE GROUPS

$\mathbb{F}_2 =$ free group on 2 generators $\{a, b\}$
 $=$ all words in a, b, a^{-1}, b^{-1} with cancellations.

$L^2(\mathbb{F}_2) = \{f : \sum_{x \in \mathbb{F}_2} |f(x)|^2 < \infty\}$, the Hilbert space of all square-integrable functions on \mathbb{F}_2 .

Each $x \in \mathbb{F}_2$ acts on $L^2(\mathbb{F}_2)$ by $(S_x(f))(y) = f(xy)$.

$L(\mathbb{F}_2) =$ von Neumann algebra generated by all S_x . It has a state $\varphi[S] = \langle S\delta_e, \delta_e \rangle$ (“ $= f(e)$ ”).

If $\varphi[p_i(a)] = 0$, $\varphi[q_i(b)] = 0$, then

$$\varphi[p_1(a)q_1(b) \dots p_k(a)q_k(b)] = 0.$$

FREE PROBABILITY THEORY

Voiculescu (~ 80): this says a, b are *freely independent*. Large independent random matrices are asymptotically freely independent.

Free probability: a non-commutative probability theory, with “independence” replaced by “free independence”. Lives “in the large N limit”, but not only there.

Many statements from probability theory have free analogs.

Theorem (Free central limit theorem). *Let X_1, X_2, \dots, X_n be freely independent with respect to a state φ , and have mean 0 and variance 1. Then*

$$\frac{X_1 + \dots + X_n}{\sqrt{n}} \xrightarrow{d} \sigma.$$

Why semicircle appears in both contexts:

$$\begin{array}{ccccccc} A_1^{(N)} & \dots & A_n^{(N)} & \sim & \frac{A_1^{(N)} + \dots + A_n^{(N)}}{\sqrt{n}} \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \dots & X_n & \sim & \frac{X_1 + \dots + X_n}{\sqrt{n}} \end{array}$$

Have analogs of

- Infinitely divisible distributions
- Convolution \boxplus and harmonic analysis
- Entropy

Connections to: combinatorics, representation theory, orthogonal polynomials, Yang-Mills theory, etc.

Another way to look at the CLT: as a fixed point theorem. Corresponding to $\frac{X_1+X_2}{\sqrt{2}}$, have the operator

$$C : \mu \mapsto (\mu \boxplus \mu) \circ S_{1/\sqrt{2}}.$$

σ is an attracting fixed point for it.

Behavior in the neighborhood of the fixed point (M.A.'99): C is a non-linear operator. Its derivative is compact, with eigenfunctions T_n^* , eigenvalues $2^{1-n/2}$. Here $T_n =$ Chebyshev polynomials of the 1st kind.

Have similar results for other free convolution semigroups (M.A.'02).

Most importantly, applications to the theory of von Neumann algebras. Example: a von Neumann algebra is *prime* is $\mathcal{A} \not\cong \mathcal{B} \otimes \mathcal{C}$ for any infinite-dimensional \mathcal{B}, \mathcal{C} .

A definition in search of an example.

Why rare: $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{A}$ common. For $\mathcal{A} = \lim M_n$,

$$\begin{array}{ccc} M_n \otimes M_n & \cong & M_{n^2} \\ \mathcal{A} \otimes \mathcal{A} & \cong & \mathcal{A} \end{array}$$

II_1 -factors: von Neumann algebras with a trace and a trivial center (\approx simple).

First known example of a prime II_1 -factor: $L(\mathbb{F}_n)$.

Still know little about $L(\mathbb{F}_n)$, for example do not know if $L(\mathbb{F}_2) \cong L(\mathbb{F}_3)$ (Kadison ~ 60). Know that either all $L(\mathbb{F}_r) \cong L(\mathbb{F}_s)$ for all $r, s \in \mathbb{R}_+ \cup \infty$, or they are all non-isomorphic.

NON-CROSSING PARTITIONS

Free independence hard to use for calculations, e.g.

$$\varphi[ab^2aba] = ?$$

Speicher '90:

$$\varphi[\cdot] = \sum_{\pi \in NC(n)} R_{\pi}.$$

$NC(n)$ = non-crossing partitions.

R = free cumulant functional, $R_{\pi} = \prod_{B \in \pi} R_B$. For example,

$$\begin{aligned} R_{(1,5)(2,3)(4)}(X_1 X_2 X_3 X_4 X_5) \\ = R(X_1 X_5) R(X_2 X_3) R(X_4). \end{aligned}$$

Here $r_k = R(a^k) =$ free cumulants of a .

Why simplifies calculations: $R(\text{free variables}) = 0$.

This implies the free independence property.

$$\varphi[a_1 b_1 a_2 b_2 \dots a_n b_n] = \sum_{\pi \in NC(2n)} R_\pi(a_1 b_1 a_2 b_2 \dots a_n b_n).$$

π connect only a 's to a 's, b 's to b 's $\Rightarrow \pi$ contains a singleton.

Each a, b centered \Rightarrow each term is 0.

Example .

$$\begin{aligned} & \varphi[ab^2aba] \\ &= R(a)^3 \varphi[b^3] + R(a^2)R(a)\varphi[b^2]\varphi[b] + R(a^3)\varphi[b^2]\varphi[b] \end{aligned}$$

Non-crossing partitions and random matrices:

$$\frac{1}{N} \mathbb{E}[\text{Tr}[A^4]] = \frac{1}{N} \sum_{i,j,k,l} \mathbb{E}[A_{ij}A_{jk}A_{kl}A_{li}]$$

Gaussian matrices: all moments expressed through the 2nd order moments.

$$\begin{aligned} & \mathbb{E} \left[\sum_{i,j,k} A_{ij}A_{ji}A_{ik}A_{ki} \right. \\ & + \sum_{i,j,k} A_{ij}A_{jk}A_{kj}A_{ji} \\ & + \sum_{i,j} A_{ii}A_{ii}A_{ij}A_{ji} \\ & \left. + \text{ more terms} \right] \end{aligned}$$

Another appearance of non-crossing partitions. Let $\{X(t)\}$ be a process. Let π be any partition, for example

$$\pi = (1, 3, 5)(2, 4, 6)$$

The corresponding stochastic measure is

$$\begin{aligned} & \text{St}_\pi(t) \\ &= \int_{[0,t]^2} dX(s_1)dX(s_2)dX(s_1)dX(s_2)dX(s_1)dX(s_2). \end{aligned}$$

Defined and investigated by Rota and Wallstrom '97 for $\{X(t)\}$ a (classical) Lévy process.

For scalar-valued measures,

$$\begin{aligned} \int_{[0,1)^2} d\mu(s)d\nu(t) \\ &= \int_0^1 \mu([0, t))d\nu(t) + \int_0^1 \nu([0, s))d\mu(s) \\ &\quad (+ \int_0^1 d\mu(s)d\nu(s)). \end{aligned}$$

If $X(t)$ are operators, $\int dX dX \neq 0$. For example,

$$\begin{aligned} &\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \\ &= \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & 1 \end{pmatrix} \right) \\ &\quad \times \left(\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & 1 \end{pmatrix} \quad (\leftrightarrow \int X(s)dX(s)) \\ &+ \begin{pmatrix} 0 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix} \quad (\leftrightarrow \int dX(s)X(s)) \\ &+ \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & 1 \end{pmatrix} \\ &\quad (\leftrightarrow \int dX(s)dX(s)) \end{aligned}$$

Instead, let $\{X(t)\}$ be a bounded free Lévy process, i.e. a stationary (operator-valued) process with freely independent increments.

Theorem . (M.A.'00) St_π are well-defined. Moreover, $St_\pi = 0$ unless π is non-crossing.

Why want St_π : can write products of multiple integrals as sums of integrals with respect to stochastic measures (Itô formulas).

Example .

$$\begin{aligned} & \left(\int dX(s)dX(t) \right) \cdot \left(\int dX(u)dX(v) \right) \\ &= St_{(1)(2)(3)(4)} + St_{(1,3)(2)(4)} + St_{(1)(2,4)(3)}. \end{aligned}$$

Relation to free cumulants: $R_\pi = \varphi[\text{St}_\pi]$.

In fact (M.A.'01) St_π can be expressed through simple multiple integrals and the free cumulants dependent on the *inner* classes of the non-crossing partition π .

Example .

$$\begin{aligned} & \text{St}_{(1)(2,6)(3,4)(5)(7,8)}(t) \\ &= \int_{[0,t]^5} dX(s_1) d\Delta_2(s_2) r_2 ds_3 r_1 ds_4 d\Delta_2(s_5), \end{aligned}$$

where $\Delta_k(t) = \int_0^t (dX(s))^k$ are the higher diagonal measures (higher variations).

Other projects:

- Relate the “linearization around the fixed point” results to fluctuation results for random matrices.
- Develop stochastic integration with respect to free processes. Partially done (M.A.'02), need more machinery, martingale inequalities etc. What algebras do these processes generate?
(Free n -dimensional Brownian motion generates $L(\mathbb{F}_n)$).
- q -interpolations between the free and the classical world. Free corresponds to $q = 0$, while $q = 1$ is symmetric and $q = -1$ is anti-symmetric. Partially done: q -Lévy processes (M.A.'01). Have an interesting relation to the theory of orthogonal polynomials in the free case (M.A.'02). Would like such a relation for other q .
- Relate the free Lévy process results to the representation theory of S_∞ (Biane).