

# GEOMETRIC GAMMA VALUES AND ZETA VALUES IN POSITIVE CHARACTERISTIC

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ABSTRACT. In analogy with values of the classical Euler  $\Gamma$ -function at rational numbers and the Riemann  $\zeta$ -function at positive integers, we consider Thakur's geometric  $\Gamma$ -function evaluated at rational arguments and Carlitz  $\zeta$ -values at positive integers. We prove that, when considered together, all of the algebraic relations among these special values arise from the standard functional equations of the  $\Gamma$ -function and from the Euler-Carlitz relations and Frobenius  $p$ -th power relations of the  $\zeta$ -function.

## 1. INTRODUCTION: A TALE OF TWO MOTIVES

The period  $\tilde{\pi}$  of the Carlitz module is central to the world of function field arithmetic. Indeed it appears in several disparate places, from explicit class field theory to Gauss sums to Drinfeld modular forms (see [11, 16]). Moreover,  $\tilde{\pi}$  is closely related to values of Thakur's geometric  $\Gamma$ -function [14] and the Carlitz  $\zeta$ -function [6, 11]. Recent work of Anderson, Brownawell, and Papanikolas [4] (on  $\Gamma$ -values) and Chang and Yu [9] (on  $\zeta$ -values) have determined all algebraic relations among special values of these functions individually.

Since  $\tilde{\pi}$  links these special values together, it is natural to ask to what extent there are algebraic relations among  $\Gamma$ -values and  $\zeta$ -values when considered together, and this is the question addressed in the present paper. The answer anticipated by the theory of  $t$ -motives is that  $\tilde{\pi}$  is the only link between  $\Gamma$ -values and  $\zeta$ -values, and indeed our theorem (Theorem 1.3.1) shows that aside from relations among  $\Gamma$ -values and  $\zeta$ -values involving  $\tilde{\pi}$  there are no other algebraic relations. In summary this is the tale of two motives, bringing the  $t$ -motive for  $\Gamma$ -values introduced in [4] and the  $t$ -motive for  $\zeta$ -values from [9] on the same stage.

**1.1. Geometric  $\Gamma$ -values.** Let  $\mathbb{F}_q$  be the finite field with  $q$  elements, where  $q$  is a power of a prime number  $p$ . Let  $A := \mathbb{F}_q[\theta]$  and  $k := \mathbb{F}_q(\theta)$ , where  $\theta$  is a variable. Let  $A_+ \subseteq A$  be the subset of monic polynomials. Let  $k_\infty := \mathbb{F}_q((1/\theta))$  be the completion of  $k$  with respect to the infinite absolute value on  $k$ , for which  $|\theta|_\infty = q$ . Let  $\mathbb{C}_\infty$  be the completion of a fixed algebraic closure  $\overline{k_\infty}$  of  $k_\infty$ , and finally let  $\overline{k}$  be the algebraic closure of  $k$  in  $\overline{k_\infty}$ .

Working in analogy with the classical Euler  $\Gamma$ -function, Thakur [14] studied the geometric  $\Gamma$ -function over  $A$ , which is a specialization of the two-variable  $\Gamma$ -function of Goss [10],

$$\Gamma(z) := \frac{1}{z} \prod_{n \in A_+} \left(1 + \frac{z}{n}\right)^{-1}, \quad z \in \mathbb{C}_\infty.$$

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It is a meromorphic function on  $\mathbb{C}_\infty$  with poles at zero and  $-n \in -A_+$  and satisfies several functional equations, which are analogous to the translation, reflection, and Gauss multiplication identities satisfied by the classical  $\Gamma$ -function.

*Special Gamma values* are those values  $\Gamma(r)$  with  $r \in k \setminus A$ . Since, when  $a \in A$ ,  $\Gamma(a)$  is either  $\infty$  or in  $k$ , we can restrict to special  $\Gamma$ -values for transcendence questions. Now the functional equations for  $\Gamma(z)$  induce families of algebraic relations among special  $\Gamma$ -values. Moreover, if for  $x, y \in \mathbb{C}_\infty^\times$  we set  $x \sim y$  when  $x/y \in \bar{k}^\times$ , then for all  $r \in k \setminus A$ ,  $a \in A$ ,  $g \in A_+$  with  $\deg g = d$ , we have the following relations:

- $\Gamma(r + a) \sim \Gamma(r)$ ,
- $\prod_{\xi \in \mathbb{F}_q^\times} \Gamma(\xi r) \sim \tilde{\pi}$ ,
- $\prod_{a \in A/g} \Gamma\left(\frac{r+a}{g}\right) \sim \tilde{\pi}^{\frac{q^d-1}{q-1}} \Gamma(r)$ .

Here  $\tilde{\pi}$  is algebraic over  $k_\infty$  and is a fundamental period for the Carlitz module, much in the same way that  $2\pi\sqrt{-1}$  is a fundamental period for the multiplicative group  $\mathbb{G}_m$  over  $\mathbb{C}$ . In analogy with the transcendence of  $2\pi\sqrt{-1}$  over  $\mathbb{Q}$ , Wade [17] proved that  $\tilde{\pi}$  is transcendental over  $k$ .

As in the classical case, natural questions arise about the transcendence and algebraic independence of these special  $\Gamma$ -values, and much is now known. As observed by Thakur [14], for  $q = 2$  all values of  $\Gamma(r)$ ,  $r \in k \setminus A$ , are  $\bar{k}$ -multiples of  $\tilde{\pi}$  and hence are transcendental over  $k$ . Thakur also related other special values to periods of Drinfeld modules and deduced their transcendence. Sinha [13] proved the first transcendence result for a general class of special  $\Gamma$ -values: he showed that  $\Gamma\left(\frac{a}{f} + b\right)$  is transcendental over  $k$  whenever  $a, f \in A_+$ ,  $\deg a < \deg f$ , and  $b \in A$ . Sinha's result was obtained by representing the  $\Gamma$ -values in question as periods of certain  $t$ -modules over  $\bar{k}$  using the soliton functions of Anderson [2] and then invoking a transcendence criterion of Gelfond-Schneider type established by Yu [19]. Expanding on Sinha's method, Brownawell and Papanikolas [5] represented all values  $\Gamma(r)$ ,  $r \in k \setminus A$ , as periods of  $t$ -modules over  $\bar{k}$  and thus proved transcendence for all special  $\Gamma$ -values.

For algebraic relations among special  $\Gamma$ -values, Thakur [14] adapted the Deligne-Koblitz-Ogus criterion to this setting and devised a diamond bracket criterion to determine which algebraic relations among special  $\Gamma$ -values arise from the functional equation relations. More specifically, a  $\Gamma$ -monomial is a monomial, with positive or negative exponents, in  $\tilde{\pi}$  and special  $\Gamma$ -values, and Thakur's criterion can decide whether a given  $\Gamma$ -monomial is in  $\bar{k}$ . In [5], Brownawell and Papanikolas showed that the only  $\bar{k}$ -linear relations among  $1$ ,  $\tilde{\pi}$ , and special  $\Gamma$ -values are those explained by the diamond bracket relations. This result was obtained by analyzing the sub- $t$ -module structure of Sinha's  $t$ -modules and then invoking Yu's sub- $t$ -module theorem [21], which plays the role here of Wüstholz's subgroup theorem [18].

In 2004, Anderson, Brownawell, and Papanikolas [4] established a new linear independence criterion (the so-called "ABP-criterion"), which is a motivic translation of Yu's sub- $t$ -module theorem. They adapted Sinha's construction to create  $t$ -motives whose periods contain the special  $\Gamma$ -values in question. Again a key component was the interpolation of special  $\Gamma$ -values via Anderson's soliton functions [2]. Anderson, Brownawell, and Papanikolas used the ABP-criterion to show that all algebraic relations over  $\bar{k}$  among special  $\Gamma$ -values arise from diamond bracket relations among  $\Gamma$ -monomials, and thus showed that all algebraic relations among special  $\Gamma$ -values can be explained by the standard functional equations. As a consequence, the transcendence degrees of the fields generated by special  $\Gamma$ -values can be obtained explicitly.

**Theorem 1.1.1** (Anderson-Brownawell-Papanikolas, [4, Cor. 1.2.2]). *For all  $f \in A_+$  of positive degree, the transcendence degree of the field*

$$\bar{k}(\{\tilde{\pi}\} \cup \{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\})$$

over  $\bar{k}$  is  $1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times$ .

**1.2. Carlitz  $\zeta$ -values.** There is also another set of special values closely related to the Carlitz period  $\tilde{\pi}$ . In [6] Carlitz considered the power sums

$$\zeta_C(n) := \sum_{a \in A_+} \frac{1}{a^n} \in k_\infty, \quad n = 1, 2, 3, \dots,$$

which are now called *Carlitz zeta values*. In analogy with values of the Riemann  $\zeta$ -function at positive even integers, Carlitz discovered that  $\zeta_C(n)/\tilde{\pi}^n$  is in  $k$  whenever  $n$  is divisible by  $q - 1$ . Thus we call a positive integer  $n$  **even** if it is a multiple of  $q - 1$ . The ratios  $\zeta_C(n)/\tilde{\pi}^n$  for **even**  $n$  involve what are now called Bernoulli-Carlitz numbers, and the theory is analogous to the case of the Riemann  $\zeta$ -function. In particular, when  $q = 2$ , all  $\zeta_C(n)$  are  $k$ -multiples of  $\tilde{\pi}^n$ .

For each positive integer  $n$ , Anderson and Thakur [3] introduced the  $n$ -th tensor power  $C^{\otimes n}$  of the Carlitz module  $C$  and explicitly related  $\zeta_C(n)$  to the last coordinate of the logarithm of a special algebraic point of  $C^{\otimes n}$ . Using this result, Yu [20] proved that each  $\zeta_C(n)$  is transcendental over  $k$ , and furthermore, when  $n$  is not a multiple of  $q - 1$ , he established the transcendence of  $\zeta_C(n)/\tilde{\pi}^n$ . Later in 1997, Yu [21] proved that the Euler-Carlitz relations are the only  $\bar{k}$ -linear relations among Carlitz  $\zeta$ -values at positive integers.

For algebraic relations among Carlitz  $\zeta$ -values, in addition to the Euler-Carlitz relations, there are also the Frobenius  $p$ -th power relations: for positive integers  $m, n$ ,

$$\zeta_C(p^m n) = \zeta_C(n)^{p^m}.$$

Chang and Yu [9] extended Yu's previous results on  $\bar{k}$ -linear relations and proved that all algebraic relations over  $\bar{k}$  among Carlitz  $\zeta$ -values at positive integers arise from the Euler-Carlitz relations and the Frobenius relations.

**Theorem 1.2.1** (Chang-Yu [9, p. 322]). *For any positive integer  $s$ , the transcendence degree of the field*

$$\bar{k}(\tilde{\pi}, \zeta_C(1), \dots, \zeta_C(s))$$

over  $\bar{k}$  is  $s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/(p(q-1)) \rfloor + 1$ .

To prove this theorem Chang and Yu adapted methods of Papanikolas [12] on algebraic independence of Carlitz logarithms to deduce algebraic independence results of Carlitz polylogarithms. In turn, they then proved the theorem by using results of Anderson and Thakur [3], who showed that Carlitz  $\zeta$ -values can be explicitly expressed in terms of  $k$ -linear combinations of Carlitz polylogarithms with algebraic arguments.

**1.3. Special  $\Gamma$ -values and Carlitz  $\zeta$ -values.** The main theorem of this paper is that, as one might expect, all the algebraic relations among special  $\Gamma$ -values and Carlitz  $\zeta$ -values arise from the standard relations. One might anticipate similar results in the classical setting, though little is known in these directions to date.

**Theorem 1.3.1** (cf. Theorem 5.1.2). *Given  $f \in A_+$  with positive degree and  $s$  a positive integer, the transcendence degree of the field*

$$\bar{k}(\{\tilde{\pi}\} \cup \{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\} \cup \{\zeta_C(1), \dots, \zeta_C(s)\})$$

over  $\bar{k}$  is

$$1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times + s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/(p(q-1)) \rfloor.$$

Our tool for proving algebraic independence is the main theorem of [12], which is an application of the ABP-criterion [4] to monomials of periods and is in some sense a function field version of Grothendieck's conjecture on periods of abelian varieties. The  $t$ -motive associated to special  $\Gamma$ -values turns out to have geometric complex multiplication from a Carlitz cyclotomic field, which enables us to show that the Galois group of this “ $\Gamma$ -motive” is a torus contained inside a finite product of the Weil restriction of scalars of  $\mathbb{G}_m$  from the cyclotomic field in question. On the other hand, according to [9], the Galois group of the “ $\zeta$ -motive,” i.e. the  $t$ -motive associated to Carlitz  $\zeta$ -values, is an extension of  $\mathbb{G}_m$  by a vector group. Once we consider the direct sum of these two  $t$ -motives, we deduce that the resulting Galois group is an extension of a torus by a vector group and compute its dimension to prove the result.

*Remark 1.3.2.* The  $\Gamma$ -function considered in this paper is “geometric” in the sense that it is defined via the theory of Carlitz cyclotomic covers of  $\mathbb{P}^1$  (see §3). In the theory of function fields there is also an “arithmetic”  $\Gamma$ -function studied by Carlitz and Goss [11], and which also has a rich transcendence theory [15]. One may ask about algebraic independence questions for the arithmetic  $\Gamma$ -function and its relations with Carlitz  $\zeta$ -values as well. These questions are addressed in [8].

## 2. GALOIS GROUPS OF $t$ -MOTIVES

### 2.1. Notation and preliminaries.

$\mathbb{F}_q$	= the finite field with $q$ elements, for $q$ a power of a prime number $p$ .
$\theta, t, z$	= independent variables.
$A$	= $\mathbb{F}_q[\theta]$ , the polynomial ring in the variable $\theta$ over $\mathbb{F}_q$ .
$A_+$	= the set of monic elements of $A$ .
$k$	= $\mathbb{F}_q(\theta)$ , the fraction field of $A$ .
$k_\infty$	= $\mathbb{F}_q((1/\theta))$ , the completion of $k$ with respect to the place at infinity.
$\overline{k_\infty}$	= a fixed algebraic closure of $k_\infty$ .
$\bar{k}$	= the algebraic closure of $k$ in $\overline{k_\infty}$ .
$\tilde{\theta}$	= a fixed choice in $\bar{k}$ of a $(q-1)$ -th root of $-\theta$ .
$\mathbb{C}_\infty$	= the completion of $\overline{k_\infty}$ with respect to the canonical extension of $\infty$ .
$ \cdot _\infty$	= the absolute value on $\mathbb{C}_\infty$ , normalized so that $ \theta _\infty = q$ .
$\mathbb{T}$	= $\{f \in \mathbb{C}_\infty[[t]] \mid f \text{ converges on }  t _\infty \leq 1\}$ , the Tate algebra over $\mathbb{C}_\infty$ .
$\mathbb{L}$	= the fraction field of $\mathbb{T}$ .
$\mathbb{G}_a$	= the additive group.
$\mathrm{GL}_{r/F}$	= for a field $F$ , the $F$ -group scheme of invertible $r \times r$ matrices.
$\mathbb{G}_m$	= $\mathrm{GL}_1$ , the multiplicative group.

For  $n \in \mathbb{Z}$ , given a Laurent series  $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$ , we define the  $n$ -fold twist of  $f$  by  $f^{(n)} = \sum_i a_i^{q^n} t^i$ . For each  $n$ , the twisting operation is an automorphism of  $\mathbb{C}_\infty((t))$  and stabilizes several subrings, e.g.,  $\bar{k}[[t]]$ ,  $\bar{k}[t]$ , and  $\mathbb{T}$ . More generally, for any matrix  $B$  with entries in  $\mathbb{C}_\infty((t))$ , we define  $B^{(n)}$  by the rule  $B^{(n)}_{ij} = B_{ij}^{(n)}$ . Also of some note (cf. [12,

Lem. 3.3.2]) is that

$$\mathbb{F}_q[t] = \{f \in \mathbb{T} \mid f^{(-1)} = f\}, \quad \mathbb{F}_q(t) = \{f \in \mathbb{L} \mid f^{(-1)} = f\}.$$

A power series  $f = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_{\infty}[[t]]$  that satisfies

$$\lim_{i \rightarrow \infty} \sqrt[i]{|a_i|_{\infty}} = 0, \quad [k_{\infty}(a_0, a_1, a_2, \dots) : k_{\infty}] < \infty,$$

is called an *entire power series*. As a function of  $t$ , such a power series  $f$  converges on all of  $\mathbb{C}_{\infty}$  and, when restricted to  $\overline{k_{\infty}}$ ,  $f$  takes values in  $\overline{k_{\infty}}$ . The ring of entire power series is denoted by  $\mathbb{E}$ , and it is invariant under  $n$ -fold twisting  $f \mapsto f^{(n)}$ .

**2.2. Galois groups and Frobenius difference equations.** We follow [12] (see also [1, 4]) in working with  $t$ -motives and their Galois groups. Let  $\overline{k}[t, \sigma]$  be the polynomial ring in variables  $t$  and  $\sigma$  subject to the relations,

$$ct = tc, \quad \sigma t = t\sigma, \quad \sigma c = c^{1/q}\sigma, \quad c \in \overline{k}.$$

Thus for  $f \in \overline{k}[t]$ , one has  $\sigma f = f^{(-1)}\sigma$ . An *Anderson  $t$ -motive* is a left  $\overline{k}[t, \sigma]$ -module  $\mathcal{M}$ , which is free and finitely generated both as a left  $\overline{k}[t]$ -module and as a left  $\overline{k}[\sigma]$ -module and which satisfies

$$(t - \theta)^N \mathcal{M} \subseteq \sigma \mathcal{M},$$

for integers  $N$  sufficiently large. The ring of Laurent polynomials in  $\sigma$  with coefficients in  $\overline{k}(t)$  is denoted  $\overline{k}(t)[\sigma, \sigma^{-1}]$ . A *pre- $t$ -motive* is a left  $\overline{k}(t)[\sigma, \sigma^{-1}]$ -module that is finite dimensional over  $\overline{k}(t)$ . The category of pre- $t$ -motives is abelian. Moreover, there is a natural functor from the category of Anderson  $t$ -motives to the category of pre- $t$ -motives,

$$\mathcal{M} \mapsto M := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M},$$

where  $\sigma$  acts diagonally on  $M$ .

In what follows we are interested in pre- $t$ -motives  $M$  that are *rigid analytically trivial*. To define this, we let  $\mathbf{m} \in \text{Mat}_{r \times 1}(M)$  be a  $\overline{k}(t)$ -basis of  $M$ , and so multiplication by  $\sigma$  on  $M$  is represented by

$$\sigma(\mathbf{m}) = \Phi \mathbf{m},$$

for some matrix  $\Phi \in \text{GL}_r(\overline{k}(t))$ . Note that if  $M$  arises from an Anderson  $t$ -motive  $\mathcal{M}$  and  $\mathbf{m} \in \text{Mat}_{r \times 1}(\mathcal{M})$ , then necessarily also  $\Phi \in \text{Mat}_r(\overline{k}[t])$ . Now  $M$  is said to be rigid analytically trivial if there exists  $\Psi \in \text{GL}_r(\mathbb{L})$  so that

$$\sigma(\Psi) = \Psi^{(-1)} = \Phi \Psi.$$

The existence of such a matrix  $\Psi$  is equivalent to the natural map of  $\mathbb{L}$ -vector spaces,

$$\mathbb{L} \otimes_{\mathbb{F}_q(t)} M^{\text{B}} \rightarrow M^{\dagger},$$

being an isomorphism, where

$$M^{\dagger} := \mathbb{L} \otimes_{\overline{k}(t)} M, \text{ on which } \sigma \text{ acts diagonally,}$$

$$M^{\text{B}} := \text{the } \mathbb{F}_q(t)\text{-submodule of } M^{\dagger} \text{ fixed by } \sigma, \text{ the 'Betti cohomology' of } M.$$

We then call  $\Psi$  a rigid analytic trivialization for the matrix  $\Phi$ , and in this situation  $\Psi^{-1}\mathbf{m}$  is an  $\mathbb{F}_q(t)$ -basis of  $M^{\text{B}}$ .

Rigid analytically trivial pre- $t$ -motives form a neutral Tannakian category over  $\mathbb{F}_q(t)$  with fiber functor  $M \mapsto M^{\text{B}}$  [12, Thm. 3.3.15]. The strictly full Tannakian subcategory generated by the images of Anderson  $t$ -motives is called the category of  *$t$ -motives*, which we denote by  $\mathcal{T}$ . By Tannakian duality, for each  $t$ -motive  $M$  of dimension  $r$  over  $\overline{k}(t)$ , the Tannakian

subcategory  $\mathcal{T}_M$  generated by  $M$  is equivalent to the category of finite dimensional representations over  $\mathbb{F}_q(t)$  of some algebraic group  $\Gamma_M \subseteq \mathrm{GL}_r/\mathbb{F}_q(t)$ . This algebraic group is called the *Galois group* of the  $t$ -motive  $M$ . Note that in this situation we always have a faithful representation

$$\varphi : \Gamma_M \hookrightarrow \mathrm{GL}(M^{\mathbb{B}}),$$

which is called the tautological representation of  $M$ . The following theorem connects Galois groups of  $t$ -motives to the transcendence degrees of interest.

**Theorem 2.2.1** (Papanikolas [12, Thm. 1.1.7]). *Let  $M$  be a  $t$ -motive with Galois group  $\Gamma_M$ . Suppose that  $\Phi \in \mathrm{GL}_r(\bar{k}(t)) \cap \mathrm{Mat}_r(\bar{k}[t])$  represents multiplication by  $\sigma$  on  $M$  and that  $\det \Phi = c(t - \theta)^s$ ,  $c \in \bar{k}^\times$ . Let  $\Psi \in \mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_r(\mathbb{E})$  be a rigid analytic trivialization of  $\Phi$ . Then the transcendence degree of the field*

$$\bar{k}(\Psi_{ij}(\theta) \mid 1 \leq i, j \leq r)$$

over  $\bar{k}$  is equal to the dimension of  $\Gamma_M$ .

Papanikolas [12] has further connected these Galois groups to the Galois theory of Frobenius difference equations, in analogy with classical differential Galois theory. This provides a method for the explicit computation of  $\Gamma_M$  for a  $t$ -motive  $M$ . Assume that  $\Phi$  and  $\Psi$  are chosen for  $M$  as above, and let  $\Psi_1, \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$  be the matrices such that  $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$  and  $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$ . Let  $\tilde{\Psi} := \Psi_1^{-1} \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L})$ . Now let  $X := (X_{ij})$  be an  $r \times r$  matrix of independent variables. We define an  $\mathbb{F}_q(t)$ -algebra homomorphism

$$\mu_\Psi : \mathbb{F}_q(t)[X, 1/\det X] \rightarrow \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L},$$

by setting  $\mu_\Psi(X_{ij}) = \tilde{\Psi}_{ij}$ . Finally we set

$$(1) \quad \Gamma_\Psi := \mathrm{Spec} \mathrm{Im} \mu_\Psi.$$

Letting  $\Lambda_\Psi$  be the field generated over  $\bar{k}(t)$  by the entries of  $\Psi$ , we obtain the following results.

**Theorem 2.2.2** (Papanikolas [12, Thms. 4.2.11, 4.3.1, 4.5.10]). *Let  $M$  be a  $t$ -motive with Galois group  $\Gamma_M$ . Let  $\Phi \in \mathrm{GL}_r(\bar{k}(t))$  represent multiplication by  $\sigma$  on  $M$ , and let  $\Psi$  be a rigid analytic trivialization for  $\Phi$ . Then*

- (a)  $\Gamma_\Psi$  is an affine algebraic group scheme over  $\mathbb{F}_q(t)$ ,
- (b)  $\Gamma_\Psi$  is absolutely irreducible and smooth over  $\mathbb{F}_q(t)$ ,
- (c)  $\dim \Gamma_\Psi = \mathrm{tr. deg}_{\bar{k}(t)} \Lambda_\Psi$ ,
- (d)  $\Gamma_\Psi$  is isomorphic to  $\Gamma_M$  over  $\mathbb{F}_q(t)$ .

*Remark 2.2.3.* Using the formalism of Frobenius difference equations, the tautological representation of  $M$  can be described as follows. For any  $\mathbb{F}_q(t)$ -algebra  $R$ , the map  $\varphi : \Gamma_M(R) \rightarrow \mathrm{GL}(R \otimes_{\mathbb{F}_q(t)} M^{\mathbb{B}})$  is given by

$$\gamma \mapsto (1 \otimes \Psi^{-1} \mathbf{m} \mapsto (\gamma^{-1} \otimes 1) \cdot (1 \otimes \Psi^{-1} \mathbf{m})).$$

**2.3. Carlitz theory and  $t$ -motives.** Put  $D_0 := 1$  and set

$$D_i = \prod_{j=0}^{i-1} (\theta^{q^i} - \theta^{q^j}), \quad i = 1, 2, \dots$$

The Carlitz exponential is defined by the power series

$$\exp_C(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i},$$

and as a function on  $\mathbb{C}_\infty$  it is an entire function and satisfies the functional equation

$$\exp_C(\theta z) = \theta \exp_C(z) + \exp_C(z)^q.$$

Moreover, one has the product expansion

$$\exp_C(z) = z \prod_{0 \neq a \in A} \left(1 - \frac{z}{a\tilde{\pi}}\right),$$

where

$$\tilde{\pi} = \theta \tilde{\theta} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1},$$

is the fundamental period of Carlitz [11, Ch. 3]. Set  $L_0 := 1$  and

$$L_i := \prod_{j=1}^i (\theta - \theta^{q^j}), \quad i = 1, 2, \dots$$

The Carlitz logarithm is defined by the power series

$$\log_C(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{L_i},$$

which converges  $\infty$ -adically for all  $z \in \mathbb{C}_\infty$  with  $|z|_\infty < |\theta|_\infty^{\frac{q}{q-1}}$ . It satisfies the functional equation

$$\theta \log_C(z) = \log_C(\theta z) + \log_C(z^q),$$

whenever the values in question are defined. As formal power series  $\exp_C(z)$  and  $\log_C(z)$  are inverses of each other with respect to composition.

For a positive integer  $n$ , the  $n$ -th Carlitz polylogarithm is defined by the series

$$(2) \quad \log_C^{[n]}(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{L_i^n},$$

which converges  $\infty$ -adically for all  $z \in \mathbb{C}_\infty$  with  $|z|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ . It can be checked that  $\log_C^{[n]}(z)$  is injective on its domain of convergence.

To connect the Carlitz theory to  $t$ -motives, we consider

$$\Omega(t) := \tilde{\theta}^{-q} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{E}.$$

It satisfies the functional equation

$$(3) \quad \Omega^{(-1)} = (t - \theta)\Omega$$

and  $\Omega(\theta) = -1/\tilde{\pi}$  (see [4, §5.1] for details). The Carlitz motive  $C$  is the  $t$ -motive with underlying vector space  $\bar{k}(t)$  itself and with  $\sigma$ -action given by

$$\sigma g = (t - \theta)g^{(-1)}, \quad g \in \bar{k}(t).$$

Thus in this case  $r = 1$  and  $\Phi = t - \theta$ , and  $\Omega$  provides a rigid analytic trivialization by (3). For  $n \geq 1$ , the  $n$ -th tensor power  $C^{\otimes n} := C \otimes_{\bar{k}(t)} \cdots \otimes_{\bar{k}(t)} C$  of the Carlitz motive also has

$\bar{k}(t)$  as its underlying space, but with  $\sigma$ -action  $\sigma g = (t - \theta)^n g^{(-1)}$ , for  $g \in \bar{k}(t)$ . Thus  $\Omega^n$  is a rigid analytic trivialization of  $C^{\otimes n}$ . By Theorem 2.2.2, it follows that the Galois group of  $C^{\otimes n}$  is  $\mathbb{G}_m$  over  $\mathbb{F}_q(t)$ , since  $\Omega$  is transcendental over  $\bar{k}(t)$ .

Finally we note that the Carlitz polylogarithms are also specialization at  $t = \theta$  of elements from  $\mathbb{T}$ . To see this, fix a positive integer  $n$ , and choose any  $\alpha \in \bar{k}^\times$  with  $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ . We consider the power series

$$(4) \quad L_{\alpha,n}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t - \theta q)^n (t - \theta q^2)^n \cdots (t - \theta q^i)^n} \in \mathbb{T}.$$

Specializing at  $t = \theta$ , we see that  $L_{\alpha,n}(\theta)$  is exactly the polylogarithm  $\log_C^{[n]}(\alpha)$  by (2). We observe that  $L_{\alpha,n}$  satisfies the functional equation,

$$(5) \quad (\Omega^n L_{\alpha,n})^{(-1)} = \alpha^{(-1)} (t - \theta)^n \Omega^n + \Omega^n L_{\alpha,n},$$

which is key for defining the the  $\zeta$ -motives in §4.2.

### 3. SPECIAL $\Gamma$ -VALUES AND $t$ -MOTIVES

Throughout this section we fix  $f \in A_+$  with positive degree, and we let  $\ell$  be the cardinality of  $(A/f)^\times$ . We review here information about the  $t$ -motives associated to special  $\Gamma$ -values from [4, §5–6], and at the end we describe their Galois groups. We apologize in advance for the confluence of notation in  $\Gamma$ ,  $\Gamma_M$ , and  $\Gamma_\Psi$ , so throughout we attempt to be as clear as possible as to which “ $\Gamma$ ” we are using.

**3.1. Carlitz cyclotomic covers.** For  $x \in k_\infty$ , we let  $\mathbf{e}(x) := \exp_C(\tilde{\pi}x)$ , and as such  $\mathbf{e} : k_\infty \rightarrow k_\infty(\theta)$ . As is well-known,  $k(\mathbf{e}(1/f))$  is a Galois extension of  $k$  with Galois group  $(A/f)^\times$ , and it is the Carlitz analogue of a cyclotomic extension of  $\mathbb{Q}$ . The Galois action of  $(A/f)^\times$  on  $k(\mathbf{e}(1/f))$  is induced by the Artin automorphism; moreover, for  $a \in A$  relatively prime to  $f$ , the action of  $a$  on  $k(\mathbf{e}(1/f))$  is defined by

$$a : \mathbf{e}(1/f) \mapsto \mathbf{e}(a/f).$$

Now given  $a \in A$ , write  $a = \theta b + \epsilon$ , with  $b \in A$  and  $\epsilon \in \mathbb{F}_q$ , and define recursively a polynomial  $C_a(t, z) \in \mathbb{F}_q[t, z]$  by

$$C_a(t, z) := \begin{cases} 0 & \text{if } a = 0, \\ C_b(t, tz + z^q) + \epsilon z & \text{if } a \neq 0. \end{cases}$$

The polynomial  $C_a(t, z)$  is called a division polynomial of  $\mathbf{e}$ , and it has the following properties:

$$C_a(\theta, \mathbf{e}(x)) = \mathbf{e}(ax), \quad C_a(t, C_b(t, z)) = C_{ab}(t, z), \quad \forall a, b \in A, x \in k_\infty.$$

Furthermore,

$$C_f(\theta, z) = \prod_{\substack{a \in A \\ \deg a < \deg f}} (z - \mathbf{e}(a/f)).$$

Hence there is a unique factor  $C_f^*(t, z) \in \mathbb{F}_q[t, z]$  of  $C_f(t, z)$  so that

$$C_f^*(\theta, z) = \prod_{\substack{a \in A \\ \deg a < \deg f, (a, f)=1}} (z - \mathbf{e}(a/f)),$$

and we can consider  $C_f^*(t, z)$  to be the  $f$ -th cyclotomic polynomial. One finds that  $C_f(t, z)$  is irreducible in  $\mathbb{F}_q[t, z]$  and remains so in  $\bar{k}[t, z]$ ; and that the rings  $\mathbf{R}_f := \mathbb{F}_q[t, z]/(C_f^*(t, z))$  and  $\mathbf{S}_f := \bar{k}[t, z]/(C_f^*(t, z))$  are Dedekind domains.

Let  $U_f/\mathbb{F}_q$  be the affine curve whose coordinate ring is  $\mathbf{R}_f$ , and let  $X_f/\mathbb{F}_q$  be its non-singular projective model. The infinite points of  $X_f$  are defined to be the closed points in  $X_f \setminus U_f$ , and they are all  $\mathbb{F}_q$ -rational. For  $a \in A$  relatively prime to  $f$ , set

$$\xi_a := (\theta, \mathbf{e}(a/f)),$$

and note that the collection  $\{\xi_a\}$  is the collection of  $\bar{k}$ -points of  $U_f$  above the point  $t = \theta$  on the affine  $t$ -line. Let

$$\bar{U}_f := \bar{k} \times_{\mathbb{F}_q} U_f, \quad \bar{X}_f := \bar{k} \times_{\mathbb{F}_q} X_f,$$

be the scalar extensions to  $\bar{k}$ . Since  $C_f^*(t, z)$  is absolutely irreducible, we see that  $X_f(\bar{k}) = \bar{X}_f(\bar{k})$  and that  $U_f(\bar{k}) = \bar{U}_f(\bar{k})$ . For  $n \in \mathbb{Z}$ , the  $n$ -fold twisting operation extends in a natural way to  $\mathbf{S}_f$  that leaves  $\mathbf{R}_f$  fixed. This action induces an action  $x \mapsto x^{(n)}$  on  $\bar{X}_f(\bar{k})$ , which raises the coordinates of  $x$  to the  $q^n$ -th power.

**3.2. Geometric constructions for  $t$ -motives.** Let  $\mathcal{A}_f$  be the free abelian group on symbols of the form  $[x]$ , where  $x \in \frac{1}{f}A/A$ . Every  $\mathbf{a} \in \mathcal{A}_f$  has a unique expression of the form

$$\mathbf{a} = \sum_{\substack{a \in A \\ \deg a < \deg f}} m_a [a/f], \quad m_a \in \mathbb{Z}.$$

If all of the coefficients  $m_a$  are non-negative, then we say that  $\mathbf{a}$  is effective. Let  $\text{wt} : \mathcal{A}_f \rightarrow \mathbb{Z}[1/(q-1)]$  be the unique group homomorphism such that for  $x \in \frac{1}{f}A/A$ ,

$$\text{wt}[x] = \begin{cases} 0 & \text{if } x \in A, \\ \frac{1}{q-1} & \text{if } x \notin A. \end{cases}$$

For each  $a \in A$  relatively prime to  $f$ , there exists a unique automorphism  $(\mathbf{a} \mapsto a \star \mathbf{a}) : \mathcal{A}_f \rightarrow \mathcal{A}_f$  of abelian groups such that

$$a \star [x] = [ax], \quad x \in \frac{1}{f}A/A.$$

Hence  $(A/f)^\times$  acts on  $\mathcal{A}_f$  via  $\star$ . Finally we define

$$\Pi(z) := z\Gamma(z) = \prod_{a \in A_+} \left(1 + \frac{z}{a}\right)^{-1},$$

which is sometimes called the ‘‘geometric factorial’’ function, and for  $\mathbf{a} \in \mathcal{A}_f$  we define  $\Pi(\mathbf{a}) \in \mathbb{C}_\infty^\times$  so that

$$\Pi([x]) = \Pi(x), \quad x \in \frac{1}{f}A/A, \quad |x|_\infty < 1.$$

The elements of the image of  $\Pi$  on  $\mathcal{A}_f$  are called  $\Pi$ -monomials of level  $f$ .

For all  $x \in k_\infty$  and integers  $N \geq 0$ , we define the *diamond bracket* by setting,

$$\langle x \rangle_N := \begin{cases} 1 & \text{if } \inf_{a \in A} |x - a - \theta^{-N-1}|_\infty < |\theta|_\infty^{-N-1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\langle x \rangle := \sum_{N=0}^{\infty} \langle x \rangle_N.$$

The sum on the right has at most one non-zero term, and in particular converges, and the value  $\langle x \rangle$  is either 0 or 1. We can extend  $\langle \cdot \rangle_N$  to  $\mathcal{A}_f$  by setting

$$\langle [x] \rangle_N = \langle x \rangle_N, \quad x \in \frac{1}{f}A/A,$$

and set

$$\langle \mathbf{a} \rangle = \sum_{N=0}^{\infty} \langle \mathbf{a} \rangle_N, \quad \mathbf{a} \in \mathcal{A}_f.$$

Fix an effective  $\mathbf{a} \in \mathcal{A}_f$ , with  $\text{wt } \mathbf{a} > 0$ . We define effective divisors of  $\overline{X}_f$  by the formulas,

$$\xi_{\mathbf{a}} := \sum_{\substack{a \in A \\ \deg a < \deg f, (a,f)=1}} \langle a \star \mathbf{a} \rangle \cdot \xi_a, \quad W_{\mathbf{a}} := \sum_{\substack{a \in A \\ \deg a < \deg f, (a,f)=1}} \sum_{N=1}^{\infty} \langle a \star \mathbf{a} \rangle_N \cdot \sum_{i=0}^{N-1} \xi_a^{(i)}.$$

Let  $\infty_{X_f}$  be the divisor sum of the infinite points of  $\overline{X}_f$  multiplied by  $q-1$ . By [4, §6.3.9], one can use Anderson's solitons [2] to define a function  $g_{\mathbf{a}}$  on  $\overline{X}_f$ , which is a rational function that is regular on  $\overline{U}_f$  and has divisor

$$\text{div}(g_{\mathbf{a}}) = -(\text{wt } \mathbf{a}) \cdot \infty_{X_f} + \xi_{\mathbf{a}} + W_{\mathbf{a}}^{(1)} - W_{\mathbf{a}}.$$

These functions provide geometric interpolations of  $\Pi$ -monomials by way of the infinite product,

$$\Pi(a \star \mathbf{a})^{-1} = \prod_{N=1}^{\infty} g_{\mathbf{a}}^{(N)}(\xi_a),$$

which converges in  $\mathbb{C}_{\infty}$ .

Now let  $\tilde{H}(\mathbf{a})$  be the left  $\overline{k}[t, \sigma]$ -module, whose underlying  $\overline{k}[t]$ -module is  $\mathbf{S}_f$  with  $\sigma$ -action given by

$$\sigma h = g_{\mathbf{a}} h^{(-1)}, \quad h \in \tilde{H}(\mathbf{a}).$$

Define

$$H(\mathbf{a}) := H^0(\overline{U}_f, \mathcal{O}_{\overline{X}_f}(-W_{\mathbf{a}}^{(1)})) \subseteq H^0(\overline{U}_f, \mathcal{O}_{\overline{X}_f}) = \tilde{H}(\mathbf{a}),$$

which is, as an ideal of  $\mathbf{S}_f$ , projective of rank one over  $\mathbf{S}_f$  and free of rank  $\ell$  over  $\overline{k}[t]$ . The module  $H(\mathbf{a})$  is an Anderson  $t$ -motive by [4, §6.4.2]. Now since  $\mathbf{R}_f$  commutes with  $\sigma$ , it follows that there is a natural inclusion

$$\mathbf{R}_f \subseteq \text{End}_{\overline{k}[t, \sigma]}(H(\mathbf{a})).$$

The main theorem of [4] that describes the essential properties of  $H(\mathbf{a})$  and its connections to special  $\Gamma$ -values is the following.

**Theorem 3.2.1** (Anderson-Brownawell-Papanikolas [4, Prop. 6.4.4]). *Let  $\mathbf{a} \in \mathcal{A}_f$  be effective with  $\text{wt } \mathbf{a} > 0$ , and let  $H(\mathbf{a})$  be the Anderson  $t$ -motive defined above. Suppose that  $\Phi_{\mathbf{a}} \in \text{Mat}_{\ell}(\overline{k}[t])$  represents multiplication by  $\sigma$  on  $H(\mathbf{a})$  with respect to any  $\overline{k}[t]$ -basis. Then there exists a rigid analytic trivialization  $\Psi_{\mathbf{a}} \in \text{GL}_{\ell}(\mathbb{T}) \cap \text{Mat}_{\ell}(\mathbb{E})$  of  $\Phi_{\mathbf{a}}$  with the property that the sets*

$$\{\Psi_{\mathbf{a}}(\theta)_{ij} \mid i, j = 1, \dots, \ell\}, \quad \{\Pi(a \star \mathbf{a})^{-1} \mid a \in A, (a, f) = 1\},$$

*span the same  $\overline{k}$ -subspace of  $\overline{k}_{\infty}$ . In particular,  $H(\mathbf{a})$  is a rigid analytically trivial Anderson  $t$ -motive, and a transcendence basis of  $\overline{k}(\Psi_{\mathbf{a}}(\theta)_{ij} \mid i, j = 1, \dots, \ell)$  over  $\overline{k}$  can be chosen among elements of the set of  $\Pi$ -monomials  $\{\Pi(a \star \mathbf{a})^{-1} \mid a \in A, (a, f) = 1\}$  that are linearly independent over  $\overline{k}$ .*

**3.3. The  $\Gamma$ -motive  $M_f$ .** For an effective  $\mathbf{a} \in \mathcal{A}_f$  with  $\text{wt } \mathbf{a} > 0$ , we define  $M_{\mathbf{a}} := \bar{k}(t) \otimes_{\bar{k}[t]} H(\mathbf{a})$ , which is a  $t$ -motive in the sense of §2.2.

**Proposition 3.3.1.** *The Galois group  $\Gamma_{M_{\mathbf{a}}}$  is a torus over  $\mathbb{F}_q(t)$ .*

*Proof.* Let  $\Phi_{\mathbf{a}} \in \text{Mat}_{\ell}(\bar{k}[t])$  represent multiplication by  $\sigma$  on  $M_{\mathbf{a}}$  with respect to a  $\bar{k}(t)$ -basis  $\mathbf{m} \in \text{Mat}_{\ell \times 1}(M_{\mathbf{a}})$  of  $M_{\mathbf{a}}$ , which is the scalar extension of a fixed  $\bar{k}[t]$ -basis of  $H(\mathbf{a})$ . Let  $\Psi_{\mathbf{a}} \in \text{GL}_{\ell}(\mathbb{T}) \cap \text{Mat}_{\ell}(\mathbb{E})$  be a rigid analytic trivialization of  $\Phi_{\mathbf{a}}$  as in Theorem 3.2.1. Finally, let  $\text{End}(M_{\mathbf{a}})$  denote the ring of endomorphisms of  $M_{\mathbf{a}}$  as a left  $\bar{k}(t)[\sigma, \sigma^{-1}]$ -module.

For  $\mathbf{f} \in \text{End}(M_{\mathbf{a}})$  there is a matrix  $F \in \text{Mat}_{\ell}(\bar{k}(t))$  so that  $\mathbf{f}(\mathbf{m}) = F\mathbf{m}$ . Since  $\mathbf{f}\sigma = \sigma\mathbf{f}$ , we have  $\Phi_{\mathbf{a}}F = F^{(-1)}\Phi_{\mathbf{a}}$ , from which it follows that the matrix  $\Psi_{\mathbf{a}}^{-1}F\Psi_{\mathbf{a}} \in \text{Mat}_{\ell}(\mathbb{L})$  is fixed by  $\sigma$  and hence  $\Psi_{\mathbf{a}}^{-1}F\Psi_{\mathbf{a}} \in \text{Mat}_{\ell}(\mathbb{F}_q(t))$ . Therefore, we have an injective map,

$$(\mathbf{f} \mapsto \mathbf{f}^B := \Psi_{\mathbf{a}}^{-1}F\Psi_{\mathbf{a}}) : \text{End}(M_{\mathbf{a}}) \hookrightarrow \text{End}_{\mathbb{F}_q(t)}(M_{\mathbf{a}}^B) = \text{Mat}_{\ell}(\mathbb{F}_q(t)).$$

Since the tautological representation  $\varphi : \Gamma_{M_{\mathbf{a}}} \hookrightarrow \text{GL}(M_{\mathbf{a}}^B)$  is functorial in  $M_{\mathbf{a}}$  (cf. [12, Thm. 4.5.3]), it follows that for an  $\mathbb{F}_q(t)$ -algebra  $R$ , any  $\gamma \in \Gamma_{M_{\mathbf{a}}}(R)$ , and any  $\mathbf{f} \in \text{End}(M_{\mathbf{a}})$ , we have the following commutative diagram.

$$\begin{array}{ccc} R \otimes_{\mathbb{F}_q(t)} M_{\mathbf{a}}^B & \xrightarrow{\varphi(\gamma)} & R \otimes_{\mathbb{F}_q(t)} M_{\mathbf{a}}^B \\ \downarrow 1 \otimes \mathbf{f}^B & & \downarrow 1 \otimes \mathbf{f}^B \\ R \otimes_{\mathbb{F}_q(t)} M_{\mathbf{a}}^B & \xrightarrow{\varphi(\gamma)} & R \otimes_{\mathbb{F}_q(t)} M_{\mathbf{a}}^B. \end{array}$$

Therefore, we have a natural embedding

$$(6) \quad \Gamma_{M_{\mathbf{a}}}(R) \hookrightarrow \text{Cent}_{\text{GL}_{\ell}(R)}(R \otimes_{\mathbb{F}_q(t)} \text{End}(M_{\mathbf{a}})).$$

Now since  $\mathbf{R}_f \subseteq \text{End}_{\bar{k}[t, \sigma]}(H(\mathbf{a}))$ , we have

$$R_f := \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} \mathbf{R}_f \hookrightarrow \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} \text{End}_{\bar{k}[t, \sigma]}(H(\mathbf{a})) \cong \text{End}(M_{\mathbf{a}}),$$

where the right-most isomorphism is from [12, Prop. 3.4.5]. Therefore, for any  $\mathbb{F}_q(t)$ -algebra  $R$ , we have an injection

$$R \otimes_{\mathbb{F}_q(t)} R_f \hookrightarrow R \otimes_{\mathbb{F}_q(t)} \text{End}(M_{\mathbf{a}}),$$

and so from (6) we have

$$(7) \quad \Gamma_{M_{\mathbf{a}}}(R) \hookrightarrow \text{Cent}_{\text{GL}_{\ell}(R)}((R \otimes_{\mathbb{F}_q(t)} R_f)^{\times}).$$

Let  $\text{Res}_{R_f/\mathbb{F}_q(t)} \mathbb{G}_m$  be the Weil restriction of scalars for  $\mathbb{G}_m/R_f$ , and note that for any  $\mathbb{F}_q(t)$ -algebra  $R$ ,

$$(\text{Res}_{R_f/\mathbb{F}_q(t)} \mathbb{G}_m)(R) = (R \otimes_{\mathbb{F}_q(t)} R_f)^{\times}.$$

Since  $[R_f : \mathbb{F}_q(t)] = \ell$ , we see that  $\text{Res}_{R_f/\mathbb{F}_q(t)} \mathbb{G}_m$  is an  $\ell$ -dimensional maximal torus inside  $\text{GL}_{\ell/\mathbb{F}_q(t)}$ , and hence

$$\text{Cent}_{\text{GL}_{\ell}}(\text{Res}_{R_f/\mathbb{F}_q(t)} \mathbb{G}_m) = \text{Res}_{R_f/\mathbb{F}_q(t)} \mathbb{G}_m.$$

From (7) we see that  $\Gamma_{M_{\mathbf{a}}}$  is a torus.  $\square$

Let  $E_f = \bar{k}(\{\tilde{\pi}\} \cup \{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\})$ . We choose a finite subset  $B_f$  of  $\mathcal{A}_f$  whose elements are effective and of positive weight so that

$$(8) \quad E_f = \bar{k}(\cup_{\mathbf{a} \in B_f} \{\Pi(a \star \mathbf{a})^{-1} \mid a \in A, (a, f) = 1\}).$$

For each  $\mathbf{a} \in B_f$ , let  $\Phi_{\mathbf{a}}$  and  $\Psi_{\mathbf{a}}$  be given as in Theorem 3.2.1. Defining

$$M_f := \oplus_{\mathbf{a} \in B_f} M_{\mathbf{a}},$$

we see that multiplication by  $\sigma$  on  $M_f$  is represented by  $\Phi_f := \bigoplus_{\mathbf{a} \in B_f} \Phi_{\mathbf{a}}$  and has rigid analytic trivialization  $\Psi_f := \bigoplus_{\mathbf{a} \in B_f} \Psi_{\mathbf{a}}$ . Since by Proposition 3.3.1 each Galois group  $\Gamma_{M_{\mathbf{a}}}$  is a torus, we see that the Galois group  $\Gamma_{M_f}$  is also a torus because by (1) we have

$$\Gamma_{M_f} \subseteq \prod_{\mathbf{a} \in B_f} \Gamma_{M_{\mathbf{a}}}.$$

On the other hand, Theorem 3.2.1 and (8) imply that  $\bar{k}(\Psi_f(\theta)) = E_f$ , and hence by Theorems 1.1.1 and 2.2.1,

$$\dim \Gamma_{M_f} = 1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times.$$

#### 4. CARLITZ $\zeta$ -VALUES AND $t$ -MOTIVES

**4.1. Carlitz polylogarithms and  $\zeta$ -values.** Let  $L_{\alpha,n}(t)$  be the series introduced in (4). We recall the following theorem of Anderson and Thakur expressing  $\zeta_C(n)$  in terms of Carlitz polylogarithms.

**Theorem 4.1.1** (Anderson-Thakur [3, §3.7–8]). *Given any positive integer  $n$ , there is a finite sequence  $h_1^{[n]}, \dots, h_{\ell_n}^{[n]} \in k$  with  $\ell_n < \frac{nq}{q-1}$ , such that*

$$(9) \quad \zeta_C(n) = \sum_{i=0}^{\ell_n} h_i^{[n]} \log_C^{[n]}(\theta^i) = \sum_{i=0}^{\ell_n} h_i^{[n]} L_{\theta^i, n}(\theta).$$

Let  $n$  be a positive integer not divisible by  $q-1$ . We let

$$N_n := k\text{-span of } \{\tilde{\pi}^n, \log_C^{[n]}(1), \log_C^{[n]}(\theta), \dots, \log_C^{[n]}(\theta^{\ell_n})\}.$$

By (9), we have  $\zeta_C(n) \in N_n$  and  $\dim_k N_n \geq 2$  since  $\zeta_C(n)$  and  $\tilde{\pi}^n$  are linearly independent over  $k$  (as  $(q-1) \nmid n$  implies  $\tilde{\pi}^n \notin k_\infty$ ). Therefore we can pick a non-negative integer  $m_n$  with  $m_n + 2 = \dim_k N_n$  and distinct integers

$$(10) \quad 0 \leq \iota(0), \dots, \iota(m_n) \leq \ell_n$$

so that both

$$(11) \quad \{\tilde{\pi}^n, \log_C^{[n]}(\theta^{\iota(0)}), \dots, \log_C^{[n]}(\theta^{\iota(m_n)})\}, \quad \{\tilde{\pi}^n, \zeta_C(n), \log_C^{[n]}(\theta^{\iota(1)}), \dots, \log_C^{[n]}(\theta^{\iota(m_n)})\},$$

are  $k$ -bases of  $N_n$ . This is possible by Theorem 4.1.1.

**4.2. The  $\zeta$ -motive  $M_{(s)}$ .** Continue with the choices in §4.1. We consider the matrices  $\Phi_n \in \mathrm{GL}_{m_n+2}(\bar{k}(t)) \cap \mathrm{Mat}_{m_n+2}(\bar{k}[t])$ ,

$$\Phi_n := \begin{bmatrix} (t-\theta)^n & 0 & \cdots & 0 \\ \theta^{\iota(0)/q}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{\iota(m_n)/q}(t-\theta)^n & 0 & \cdots & 1 \end{bmatrix},$$

and  $\Psi_n \in \mathrm{GL}_{m_n+2}(\mathbb{T}) \cap \mathrm{Mat}_{m_n+2}(\mathbb{E})$  given by

$$\Psi_n := \begin{bmatrix} \Omega^n & 0 & \cdots & 0 \\ L_{\theta^{\iota(0)}, n} \Omega^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{\theta^{\iota(m_n)}, n} \Omega^n & 0 & \cdots & 1 \end{bmatrix}.$$

By (3) and (5), we have

$$\Psi_n^{(-1)} = \Phi_n \Psi_n.$$

Note that all of the entries of  $\Psi_n$  are in  $\mathbb{E}$  and that  $\Phi_n$  defines a  $t$ -motive  $M_n$  (see [9, Lem. A.1]).

In the context of applying Theorem 2.2.1, we see from our various choices arising from (9) that  $\zeta_C(n)/\tilde{\pi}^n$  is a  $k$ -linear combination of the entries of the first column of  $\Psi_n(\theta)$ , so in order to compare different  $\zeta$ -values simultaneously we need to consider various  $M_n$  and  $\Psi_n$  simultaneously. To do this, we fix a positive integer  $s$  and let

$$U(s) := \{n \in \mathbb{Z} \mid 1 \leq n \leq s, p \nmid n, (q-1) \nmid n\}.$$

We let  $M_{(s)}$  be the direct sum  $t$ -motive  $\bigoplus_{n \in U(s)} M_n$  and define block diagonal matrices,

$$\begin{aligned} \Phi_{(s)} &:= \bigoplus_{n \in U(s)} \Phi_n, \\ \Psi_{(s)} &:= \bigoplus_{n \in U(s)} \Psi_n. \end{aligned}$$

Then  $\Phi_{(s)}$  represents multiplication by  $\sigma$  on  $M_{(s)}$ , and  $\Psi_{(s)}$  is a rigid analytic trivialization of  $\Phi_{(s)}$ . From (1) we see that any element of  $\Gamma_{\Psi_{(s)}}$  is of the form

$$\bigoplus_{n \in U(s)} \begin{bmatrix} x^n & 0 \\ * & \text{Id}_{m_n+1} \end{bmatrix},$$

where  $\text{Id}_r$  denotes the identity matrix of size  $r$ .

Now the Carlitz motive  $C$  is a sub- $t$ -motive of  $M_1 \subseteq M_{(s)}$ , so by Tannakian duality there is a surjective map

$$(12) \quad \pi : \Gamma_{\Psi_{(s)}} \twoheadrightarrow \mathbb{G}_m.$$

Let  $\mathcal{T}_{M_{(s)}}$  and  $\mathcal{T}_C$  be the strictly full Tannakian subcategories of the category  $\mathcal{T}$  of  $t$ -motives, which are generated by  $M_{(s)}$  and  $C$  respectively. The map  $\pi$  comes from the restriction of the fiber functor of  $\mathcal{T}_{M_{(s)}}$  to  $\mathcal{T}_C$ . For any  $\gamma \in \Gamma_{\Psi_{(s)}}(\overline{\mathbb{F}_q(t)})$ , it follows from the discussion in Remark 2.2.3 that the action of  $\pi(\gamma)$  on  $\overline{\mathbb{F}_q(t)} \otimes_{\overline{\mathbb{F}_q(t)}} C^{\text{B}}$  is equal to the action of the upper left-most corner of  $\gamma$ . This implies that  $\pi$  is the projection on the upper left-most corner of elements of  $\Gamma_{\Psi_{(s)}}$ .

Let  $V_{(s)}$  be the kernel of  $\pi$ , giving an exact sequence of linear algebraic groups,

$$0 \rightarrow V_{(s)} \rightarrow \Gamma_{\Psi_{(s)}} \xrightarrow{\pi} \mathbb{G}_m \rightarrow 1.$$

We see that  $V_{(s)}$  is contained in the vector group  $G_{(s)}$ , consisting of all block diagonal matrices of the form,

$$\bigoplus_{n \in U(s)} \begin{bmatrix} 1 & 0 \\ * & \text{Id}_{m_n+1} \end{bmatrix},$$

which is a unipotent group isomorphic to the direct product  $\prod_{n \in U(s)} \mathbb{G}_a^{m_n+1}$ .

**Theorem 4.2.1** (Chang-Yu [9, Thm. 4.5, Cor. 4.6], cf. Theorem 1.2.1). *Fix a positive integer  $s$ , and let  $M_{(s)}$ ,  $\Psi_{(s)}$ ,  $V_{(s)}$ , and  $G_{(s)}$  be defined as above. Then  $V_{(s)} = G_{(s)}$ , and  $\Gamma_{M_{(s)}}$  is an extension of  $\mathbb{G}_m$  by a vector group. Hence*

$$\dim \Gamma_{\Psi_{(s)}} = 1 + \sum_{n \in U(s)} (m_n + 1).$$

In particular, the quantities among

$$\{\tilde{\pi}\} \cup_{n \in U(s)} \cup_{i=0}^{m_n} \{\log_C^{[n]}(\theta^{(i)})\}$$

are algebraically independent over  $\bar{k}$ .

*Remark 4.2.2.* When we combine Theorem 4.2.1 with the choices we made in (10)–(11), we see that the field  $\bar{k}(\Psi_{(s)}(\theta))$  has a transcendence basis  $T \cup \{\tilde{\pi}\}$  with  $T \supseteq \{\zeta_C(n) \mid n \in U(s)\}$ .

## 5. ALGEBRAIC INDEPENDENCE OF $\Gamma$ -VALUES AND $\zeta$ -VALUES

**5.1. The main theorem.** To study the algebraic relations among  $\Gamma$ -values and  $\zeta$ -values simultaneously, we now combine  $\Gamma$ -motives and  $\zeta$ -motives. Fix a positive integer  $s$ , and let  $M_{(s)}$ ,  $\Phi_{(s)}$ , and  $\Psi_{(s)}$  be defined as in §4.2.

**Theorem 5.1.1.** *Assume that  $q > 2$ . Suppose  $M_0$  is a  $t$ -motive for which multiplication by  $\sigma$  is represented by  $\Phi_0 \in \text{Mat}_\nu(\bar{k}[t])$  with  $\det \Phi_0 = c(t - \theta)^m$ ,  $c \in \bar{k}^\times$ ,  $m \geq 1$ , and suppose that  $\Psi_0 \in \text{GL}_\nu(\mathbb{T}) \cap \text{Mat}_\nu(\mathbb{E})$  is a rigid analytic trivialization of  $\Phi_0$ . Suppose further that its Galois group  $\Gamma_{\Psi_0}$  is a torus over  $\mathbb{F}_q(t)$ . Finally, let  $M := M_{(s)} \oplus M_0$ ,  $\Phi := \Phi_{(s)} \oplus \Phi_0$ , and  $\Psi := \Psi_{(s)} \oplus \Psi_0$ .*

- (a) *We have  $\dim \Gamma_M = \dim \Gamma_\Psi = \dim \Gamma_{\Psi_{(s)}} + \dim \Gamma_{\Psi_0} - 1$ .*
- (b) *If  $T \cup \{\tilde{\pi}\}$  is a transcendence basis for  $\bar{k}(\Psi_{(s)}(\theta))$  over  $\bar{k}$  and  $S$  is any transcendence basis of  $\bar{k}(\Psi_0(\theta))$  over  $\bar{k}$ , then the set  $T \cup S$  is algebraically independent over  $\bar{k}$ .*

*Proof.* By (1) we have  $\Gamma_\Psi \subseteq \Gamma_{\Psi_{(s)}} \times \Gamma_{\Psi_0}$ , so that  $\Gamma_\Psi$  is a solvable group. By Tannakian duality there is surjective morphism  $\Gamma_\Psi \twoheadrightarrow \Gamma_{\Psi_{(s)}}$ , and it follows that the unipotent radical of  $\Gamma_\Psi$  has dimension  $\geq \dim \Gamma_{\Psi_{(s)}} - 1$ , which is the dimension of the unipotent radical of  $\Gamma_{\Psi_{(s)}}$ . On the other hand, we also have a surjective morphism from  $\Gamma_\Psi$  to the torus  $\Gamma_{\Psi_0}$ , which forces the dimension of the maximal torus of  $\Gamma_\Psi$  to be at least  $\dim \Gamma_{\Psi_0}$ . Hence  $\dim \Gamma_\Psi \geq \dim \Gamma_{\Psi_{(s)}} + \dim \Gamma_{\Psi_0} - 1$ . Observing now that  $\det \Psi_0(\theta)$  is a  $\bar{k}^\times$ -multiple of  $\tilde{\pi}^{-m}$ , we have  $\tilde{\pi}^m \in \bar{k}(\Psi_0(\theta))$ . Since  $\tilde{\pi}$  is also in  $\bar{k}(\Psi_{(s)}(\theta))$ , Theorem 2.2.1 gives part (a).

Part (b) also follows from (a) by using Theorem 2.2.1. To see this,

$$\bar{k}(\Psi(\theta)) \supseteq \bar{k}(T \cup \{\tilde{\pi}\} \cup S) \supseteq \bar{k}(T \cup \{\tilde{\pi}^m\} \cup S) = \bar{k}(T \cup S).$$

However, these two containments are each of finite degree, and thus part (a) implies that  $\text{tr. deg}_{\bar{k}} \bar{k}(T \cup S) = \dim \Gamma_{\Psi_{(s)}} + \dim \Gamma_{\Psi_0} - 1 = \#T + 1 + \#S - 1 = \#T + \#S$ .  $\square$

Our main theorem is as follows.

**Theorem 5.1.2.** *Given  $f \in A_+$  with positive degree and  $s$  a positive integer, the transcendence degree of the field*

$$\bar{k}(\{\tilde{\pi}\} \cup \{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\} \cup \{\zeta_C(1), \dots, \zeta_C(s)\})$$

over  $\bar{k}$  is

$$1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times + s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/(p(q-1)) \rfloor.$$

*Proof.* When  $q = 2$ , the result is true, since each  $\Gamma(r)$ ,  $r \in k \setminus A$ , is a  $\bar{k}$ -multiple of  $\tilde{\pi}$  and each  $\zeta_C(n)$ ,  $n \geq 1$ , is a  $k$ -multiple of  $\tilde{\pi}^n$ . When  $q > 2$ , we can apply Theorem 5.1.1. Indeed, by Proposition 3.3.1 we can take  $M_0 = M_f$ , which satisfies the hypotheses of Theorem 5.1.1, and  $S$  to be a transcendence basis of  $E_f$  over  $\bar{k}$ ; by Remark 4.2.2, there is a transcendence basis  $T \cup \{\tilde{\pi}\}$  for  $\bar{k}(\Psi_{(s)}(\theta))$  over  $\bar{k}$  with  $T$  containing all  $\zeta_C(n)$ ,  $n \in U(s)$ . Since  $\#S = 1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times$  and  $\#U(s) = s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/(p(q-1)) \rfloor$ , the result follows.  $\square$

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