ALGEBRAIC RELATIONS AMONG PERIODS AND LOGARITHMS OF RANK 2 DRINFELD MODULES

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Abstract. For any rank 2 Drinfeld module ρ defined over an algebraic function field, we consider its period matrix P_ρ, which is analogous to the period matrix of an elliptic curve defined over a number field. Suppose that the characteristic of the finite field $\mathbb{F}_q$ is odd and that ρ does not have complex multiplication. We show that the transcendence degree of the field generated by the entries of P_ρ over $\mathbb{F}_q(\theta)$ is 4. As a consequence, we show also the algebraic independence of Drinfeld logarithms of algebraic functions which are linearly independent over $\mathbb{F}_q(\theta)$.

1. Introduction

This paper focuses on the algebraic independence of periods, quasi-periods, and logarithms of Drinfeld modules of rank 2, without complex multiplication, that are defined over the algebraic closure of $\mathbb{F}_q(\theta)$. By the main theorem of [12], which itself builds on results in [2], the proofs break into two parts. The first is to relate the quantities in question to special values of solutions of certain Frobenius difference equations. For periods and quasi-periods these problems are solved using Anderson generating functions together with methods inspired by [13]. For logarithms we solve difference equations related to extensions of the trivial t-motive by the t-motive associated to the Drinfeld module. The second part is to show that the Galois group of the system of difference equations has maximal dimension. The difficulty here is that these quantities can have many linear relations, and characterizing these linear relations in terms of the dimension of the Galois group is quite complicated.

Yu [19, 20] proved that all linear relations among periods and logarithms of Drinfeld modules of arbitrary rank are the ones that are expected, ultimately relying on the Sub-t-module Theorem of [20]. Using Yu's methods, Brownawell [4] extended these results to values of quasi-periodic functions. Later David and Denis [7] used Yu's theorem to show there are no quadratic relations among periods of rank 2 Drinfeld modules without complex multiplication.

1.1. **Periods and Quasi-periods of Drinfeld modules.** Let $\mathbb{F}_q$ be the finite field of q elements, where q is a power of a prime p. Let $k := \mathbb{F}_q(\theta)$ be the rational function field in a variable θ over $\mathbb{F}_q$. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of $\mathbb{F}_q((\frac{1}{\theta}))$ with respect to the non-archimedean absolute value of k for which $|\theta|_\infty = q$. Let $\mathbb{C}_\infty[\tau]$ be the twisted polynomial ring in τ over $\mathbb{C}_\infty$ subject to the relation $\tau c = c^q \tau$ for $c \in \mathbb{C}_\infty$.

Now let t be an independent variable from θ. A Drinfeld $\mathbb{F}_q[t]$-module ρ (with generic characteristic) is an $\mathbb{F}_q$-algebra homomorphism $\rho : \mathbb{F}_q[t] \to \mathbb{C}_\infty[\tau]$ such that for all $a \in \mathbb{F}_q[t]$, the constant term of ρ_a, as a polynomial in τ, is $a(\theta)$. We require further that the image of

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ρ is not contained in $\mathbb{C}_\infty$. Given a subfield L of $\mathbb{C}_\infty$ containing k, we say that ρ is defined over L if all the coefficients of ρ_t fall in L. The degree of ρ_t in τ is called the rank of ρ.

The period lattice of a rank r Drinfeld $\mathbb{F}_q[t]$-module ρ is defined as follows. The exponential function of ρ is the function $\exp_\rho(z) = z + \sum_{i=1}^{\infty} \alpha_i z^{q^i}$, with $\alpha_i \in \mathbb{C}_\infty$, such that $\exp_\rho(\theta z) = \rho_t(\exp_\rho(z))$. It can be shown that $\exp_\rho$ is entire, $\mathbb{F}_q$-linear, and surjective, and it is unique. Let Λ_ρ be the kernel of $\exp_\rho$. One finds that Λ_ρ is a discrete, free $\mathbb{F}_q[\theta]$-module of rank r inside $\mathbb{C}_\infty$. We call Λ_ρ the period lattice of ρ, and any element of Λ_ρ is called a period of ρ. In 1986, Yu [18] established a fundamental theorem which asserts that any non-zero period of ρ is transcendental over k if ρ is defined over $\overline{k}$.

In analogy with de Rham cohomology for elliptic curves, in the late 1980's Anderson, Deligne, Gekeler, and Yu developed a de Rham cohomology theory for Drinfeld modules (for more details, see [5, 9, 15, 19]). Fix a rank r Drinfeld $\mathbb{F}_q[t]$-module ρ. An $\mathbb{F}_q$-linear map $\delta : \mathbb{F}_q[t] \to \mathbb{C}_\infty[\tau]\tau$ is called a biderivation if $\delta_{ab} = a(\theta)\delta_b + \delta_a \rho_b$ for any $a, b \in \mathbb{F}_q[t]$. A biderivation δ is uniquely determined by δ_t, and hence the set of all biderivations, denoted by $D(\rho)$, is a $\mathbb{C}_\infty$-vector space. Moreover, for $m \in \mathbb{C}_\infty[\tau]$ the map $\delta^{(m)} : a \mapsto a(\theta)m - m\rho_a$ is also a biderivation. The biderivation $\delta^{(m)}$ is called an inner biderivation, and we denote by $D_i(\rho)$ the set of all inner biderivations. In the case that $m \in \mathbb{C}_\infty[\tau]\tau$, $\delta^{(m)}$ is called strictly inner, and the set of all strictly inner biderivations is denoted by $D_{si}(\rho)$. The de Rham cohomology of ρ is defined to be the $\mathbb{C}_\infty$-vector space $H_{DR}(\rho) := D(\rho)/D_{si}(\rho)$.

Given $\delta \in D(\rho)$, there is a unique entire function F_δ of the form $F_\delta(z) = \sum_{i=1}^{\infty} c_i z^{q^i}$ satisfying the functional equation

$$F_\delta(a(\theta)z) - a(\theta)F_\delta(z) = \delta_a(\exp_\rho(z)), \tag{1}$$

for every $a \in \mathbb{F}_q[t]$. We call F_δ the quasi-periodic function of ρ associated to δ. It is $\mathbb{F}_q$-linear, and furthermore $F_\delta|_{\Lambda_\rho} : \Lambda_\rho \to \mathbb{C}_\infty$ is $\mathbb{F}_q[\theta]$-linear.

For the inner biderivation $\delta^{(1)} : a \mapsto a(\theta) - \rho_a$, one has $F_{\delta^{(1)}} = z - \exp_\rho(z)$ and hence $F_{\delta^{(1)}}(\lambda) = \lambda$ for $\lambda \in \Lambda_\rho$. We think of scalar multiples of $\delta^{(1)}$ as differentials of the first kind. If $\delta \notin D_i(\rho)$, the values $F_\delta(\lambda)$ for $\lambda \in \Lambda_\rho$ are called quasi-periods of ρ. For any $\delta \in D_{si}(\rho)$ one has $F_\delta(\lambda) = 0$ for $\lambda \in \Lambda_\rho$. Thus there is a well-defined map of $\mathbb{C}_\infty$-vector spaces,

$$H_{DR}(\rho) \to \mathrm{Hom}_{\mathbb{F}_q[\theta]}(\Lambda_\rho, \mathbb{C}_\infty),$$
$$\delta \mapsto (\lambda \mapsto F_\delta(\lambda)),$$

which is an isomorphism by Gekeler [9] (and Anderson, in unpublished work, using different techniques). In particular, $\dim_{\mathbb{C}_\infty} H_{DR}(\rho) = r$.

In 1990, Yu [19] established a fundamental theorem concerning the transcendence of quasi-periods. The theorem asserts that if ρ is defined over $\overline{k}$, then for any $\delta \in D(\rho) \setminus D_{si}(\rho)$ defined over $\overline{k}$ and for $0 \neq \lambda \in \Lambda_\rho$, $F_\delta(\lambda)$ is transcendental over k. When we think of $\delta \in D(\rho) \setminus D_i(\rho)$ as a differential of the second kind, Yu's theorem parallels the classical work of Schneider on elliptic integrals of the second kind.

The set $\mathrm{End}(\rho) := \{z \in \mathbb{C}_\infty : z\Lambda_\rho \subseteq \Lambda_\rho\}$ is the endomorphism ring of ρ, which is a finite $\mathbb{F}_q[\theta]$-algebra of rank at most r over $\mathbb{F}_q[\theta]$. If $[\mathrm{End}(\rho) : \mathbb{F}_q[\theta]] = r$, then ρ is said to have complex multiplication. We let K_ρ denote the fraction field of $\mathrm{End}(\rho)$.

We now fix a rank 2 Drinfeld $\mathbb{F}_q[t]$-module ρ defined over $\overline{k}$ by $\rho_t = \theta + \kappa\tau + \Delta\tau^2$, and we fix a basis $\{\omega_1, \omega_2\}$ of Λ_ρ over $\mathbb{F}_q[\theta]$. We denote by F_τ the quasi-periodic function of ρ associated to the biderivation given by $t \mapsto \tau$. The matrix

$$P_\rho := \begin{bmatrix} \omega_1 & -F_\tau(\omega_1) \\ \omega_2 & -F_\tau(\omega_2) \end{bmatrix} \tag{2}$$

is called the period matrix of ρ. As in [5, 15, 19], it is possible to identify the columns of ${P_\rho}^{\mathrm{tr}}$ as generators of the period lattice of the 2-dimensional t-module that is the extension of ρ by the additive group $\mathbb{G}_a$ and which is associated to the biderivation $t \mapsto \tau$. However, we do not pursue this further in this paper. Anderson proved an analogue of Legendre's relation for periods and quasi-periods of elliptic curves over $\mathbb{C}$, namely

$$\omega_1 F_\tau(\omega_2) - \omega_2 F_\tau(\omega_1) = \frac{\widetilde{\pi}}{\sqrt[q-1]{-\Delta}},$$

for some $(q-1)$-st root of $-\Delta$ (cf. [15, Thm. 6.4.6]). Here $\widetilde{\pi}$ is a generator of the period lattice of the Carlitz $\mathbb{F}_q[t]$-module C given by $C_t = \theta + \tau$.

Now let $\overline{k}(P_\rho)$ be the field generated by the entries of P_ρ. Thiery [16] has shown that $\mathrm{tr.deg}_{\overline{k}}\, \overline{k}(P_\rho) = 2$ if ρ has complex multiplication. When ρ does not have complex multiplication Brownawell [4] used Yu's Sub-t-module theorem [20] to prove an analogue of Masser's work on linear independence of periods and quasi-periods for elliptic curves without complex multiplication: i.e., the 6 quantities

$$1, \widetilde{\pi}, \omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)$$

are linearly independent over $\overline{k}$. The first main result of the present paper is as follows (later stated as Theorem 3.4.1).

Theorem 1.1.1. *Suppose that p is odd. Let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined over $\overline{k}$ without complex multiplication. Then the 4 quantities*

$$\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)$$

are algebraically independent over $\overline{k}$.

The omitted case of $p = 2$ imposes some added difficulties, which require further investigation (see Remark 3.3.3).

1.2. **Algebraic independence of Drinfeld logarithms.** In [20], Yu proved the full analogue of the qualitative form of Baker's theorem on logarithms of algebraic numbers for Drinfeld modules. Using Yu's Sub-t-module Theorem, Brownawell [4] proved linear independence of quasi-periods as well.

Theorem 1.2.1 (Brownawell [4, Prop. 2]; Yu [20, Thm. 4.3])**.** *Let ρ be a Drinfeld $\mathbb{F}_q[t]$-module defined over $\overline{k}$. Suppose $F_{\delta_1}, \dots, F_{\delta_s}$ are quasi-periodic functions for biderivations $\delta_1, \dots, \delta_s$ in $D(\rho)$ that are defined over $\overline{k}$ and $\mathbb{C}_\infty$-linearly independent modulo $D_i(\rho)$.*

Let $\lambda_1, \dots, \lambda_m$ be elements of $\mathbb{C}_\infty$ such that $\exp_\rho(\lambda_i) \in \overline{k}$ for $i = 1, \dots, m$. If $\lambda_1, \dots, \lambda_m$ are linearly independent over K_ρ, then the $1+m(s+1)$ quantities 1, $\lambda_1, \dots, \lambda_m$, $\cup_{j=1}^{s} \{F_{\delta_j}(\lambda_1), \dots, F_{\delta_j}(\lambda_m)\}$, are linearly independent over $\overline{k}$.

In analogy with a classical conjecture concerning the algebraic independence of logarithms of algebraic numbers, among the Drinfeld logarithms of algebraic functions one expects conjecturally that linear relations over the multiplication ring of the Drinfeld module in question are the only algebraic relations.

Conjecture 1.2.2 (Brownawell-Yu; see [3, p. 323])**.** *Under the hypotheses of Theorem 1.2.1, the $m(s+1)$ quantities $\lambda_1, \dots, \lambda_m$, $\cup_{j=1}^{s} \{F_{\delta_j}(\lambda_1), \dots, F_{\delta_j}(\lambda_m)\}$ are algebraically independent over $\overline{k}$.*

The second author has shown that the conjecture holds for the Carlitz module (since $D(C)/D_i(C) = 0$).

Theorem 1.2.3 (Papanikolas [12, Thm. 1.2.6]). *Let C be the Carlitz $\mathbb{F}_q[t]$-module defined by $C_t := \theta + \tau$. Let $\lambda_1, \dots, \lambda_m$ be elements of $\mathbb{C}_\infty$ so that $\exp_C(\lambda_i) \in \overline{k}$ for $i = 1, \dots, m$. If $\lambda_1, \dots, \lambda_m$ are linearly independent over k, then they are algebraically independent over $\overline{k}$.*

In the case of odd characteristic, using Theorem 1.1.1 we prove the above conjecture for logarithms of certain rank 2 Drinfeld modules (later stated as Theorem 4.3.3).

Theorem 1.2.4. *Suppose that p is odd. Let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined over $\overline{k}$ without complex multiplication. Let $\lambda_1, \dots, \lambda_m$ be elements of $\mathbb{C}_\infty$ such that $\exp_\rho(\lambda_i) \in \overline{k}$ for $i = 1, \dots, m$. If $\lambda_1, \dots, \lambda_m$ are linearly independent over k, then the $2m$ quantities,*

$$\lambda_1, \dots, \lambda_m, F_\tau(\lambda_1), \dots, F_\tau(\lambda_m),$$

are algebraically independent over $\overline{k}$.

It should be noted that Denis [8, Thm. 2] has shown that at least 2 of the $2m$ quantities in the theorem are algebraically independent over $\overline{k}$.

1.3. **Periods of elliptic curves and elliptic logarithms.** One can compare these results to classical conjectures about elliptic curves. Specifically, let E be an elliptic curve over $\overline{\mathbb{Q}}$. Let $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a period lattice in $\mathbb{C}$ for a given Weierstrass model of E, and let $\wp$ be the Weierstrass $\wp$-function associated to Λ. One has the Weierstrass ζ-function satisfying $\zeta'(z) = -\wp(z)$. Then each $\eta_i := 2\zeta(\frac{1}{2}\omega_i)$, for $i = 1, 2$, is called a quasi-period of E. The matrix

$$P_E := \begin{bmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{bmatrix}$$

is called the period matrix of E and the Legendre relation says that $\det P_E = \pm 2\pi\sqrt{-1}$. Conjecturally, one expects

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(P_E) = \begin{cases} 4 & \text{if } \operatorname{End}(E) = \mathbb{Z}, \\ 2 & \text{if } \operatorname{End}(E) \neq \mathbb{Z}. \end{cases}$$

This conjecture is known, by work of Chudnovsky, if E has complex multiplication, but in the non-CM case, one only knows, by work of Masser, that the periods and quasi-periods are linearly independent over $\overline{\mathbb{Q}}$.

Furthermore, one can conjecture results on elliptic logarithms. That is, suppose $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ satisfy $\wp(\lambda_i) \in \overline{\mathbb{Q}}$. If $\lambda_1, \dots, \lambda_m$ are linearly independent over the multiplication ring of E, then one expects that the $2m$ quantities,

$$\lambda_1, \dots, \lambda_m, \zeta(\lambda_1), \dots, \zeta(\lambda_m),$$

are algebraically independent over $\overline{\mathbb{Q}}$. However, the best known results involve only linear independence over $\overline{\mathbb{Q}}$, due to Masser (elliptic logarithms in the CM case), Bertrand-Masser (elliptic logarithms in the non-CM case), and Wüstholz (elliptic integrals of both the first and second kind). See [17, §4] for more details.

1.4. **Outline of the paper.** Our main tool for proving algebraic independence is rooted in a linear independence criterion of Anderson, Brownawell, and the second author [2]. This criterion is key for proving an algebraic independence theorem of the second author [12], which relates Galois groups of difference equations with algebraic relations among their specializations at $t = \theta$. The basic structure in this theory is the notion of a t-motive introduced by Anderson [1]. We emphasize that we have aimed to make all results as explicit as possible.

We review the related background in §2. We use Anderson generating functions to give explicit solutions of the difference equations associated to a given Drinfeld module, following a method of Pellarin [13]. By unpublished work of Anderson, such solutions are necessarily connected to the period matrix of the Drinfeld module, and we observe this relationship and provide full details in this situation. Hence by the main theorem of [12], Theorem 1.1.1 is reduced to calculating the dimension of the Galois group in question.

The proof of Theorem 1.1.1 is in §3. First we consider the t-motive of any rank 2 Drinfeld module, and then assuming the characteristic is odd, we establish that the tautological representation of its Galois group is semisimple. We also establish a t-motivic version of Tate's conjecture. With these properties in hand, we are able to show that if the given Drinfeld module is without complex multiplication, then its tautological Galois representation is absolutely irreducible. It follows that the motivic Galois group in question can be calculated to be GL_2, and hence Theorem 1.1.1 is proved.

Finally, the proof of Theorem 1.2.4 occupies §4. We construct a suitable t-motive such that its period matrix is related to the logarithms of the algebraic functions in question. Using the results for Theorem 1.1.1, we show that the Galois group in the case of logarithms is an extension of GL_2 by a vector group. By calculating its dimension, we prove Theorem 1.2.4.

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2. Periods and t-motives

2.1. **Notation and Preliminaries.**

$\mathbb{F}_q$	$=$ the finite field of q elements, where q is a power of a prime number p.
k	$= \mathbb{F}_q(\theta) =$ the rational function field in a variable θ over $\mathbb{F}_q$.
k_∞	$= \mathbb{F}_q((\frac{1}{\theta}))$, the completion of k with respect to the place at infinity.
$\overline{k_\infty}$	$=$ a fixed algebraic closure of k_∞.
$\overline{k}$	$=$ the algebraic closure of k in $\overline{k_\infty}$.
$\mathbb{C}_\infty$	$=$ the completion of $\overline{k_\infty}$ with respect to the canonical extension of the place at infinity.
$\lvert\cdot\rvert_\infty$	$=$ a fixed absolute value for the completed field $\mathbb{C}_\infty$ with $\lvert\theta\rvert_\infty = q$.
$\mathbb{T}$	$= \{f \in \mathbb{C}_\infty[[t]] : f$ converges on $\lvert t\rvert_\infty \le 1\}$ (the Tate algebra of $\mathbb{C}_\infty$).
$\mathbb{L}$	$=$ the fraction field of $\mathbb{T}$.
$\mathrm{GL}_{r/F}$	$=$ for a field F, the F-group scheme of invertible $r \times r$ matrices.
$\mathbb{G}_m$	$= \mathrm{GL}_1 =$ the multiplicative group.

For $n \in \mathbb{Z}$, given a Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$ we define the n-fold twist of f by the rule $f^{(n)} := \sum_i a_i^{q^n} t^i$. For each n, the twisting operation is an automorphism of the Laurent series field $\mathbb{C}_\infty((t))$ stabilizing several subrings, e.g., $\overline{k}[[t]]$, $\overline{k}[t]$ and $\mathbb{T}$. More generally, for any matrix B with entries in $\mathbb{C}_\infty((t))$ we define $B^{(n)}$ by the rule ${B^{(n)}}_{ij} := B_{ij}^{(n)}$.

A power series $f = \sum_{i=0}^\infty a_i t^i \in \mathbb{C}_\infty[[t]]$ that satisfies

$$\lim_{i\to\infty} \sqrt[i]{\lvert a_i\rvert_\infty} = 0 \quad \text{and} \quad [k_\infty(a_0, a_1, a_2, \ldots) : k_\infty] < \infty$$

is called an entire power series. As a function of t, such a power series f converges on all of $\mathbb{C}_\infty$ and, when restricted to $\overline{k_\infty}$, f takes values in $\overline{k_\infty}$. The ring of entire power series is denoted by $\mathbb{E}$.

2.2. **Tannakian category of t-motives.** In this section we follow [12] for background and terminology of t-motives. Let $\overline{k}(t)[\sigma,\sigma^{-1}]$ be the noncommutative ring of Laurent polynomials in σ with coefficients in $\overline{k}(t)$, subject to the relation

$$\sigma f = f^{(-1)}\sigma, \quad \forall\, f \in \overline{k}(t).$$

Let $\overline{k}[t,\sigma]$ be the noncommutative subring of $\overline{k}(t)[\sigma,\sigma^{-1}]$ generated by t and σ over $\overline{k}$. An Anderson t-motive is a left $\overline{k}[t,\sigma]$-module $\mathcal{M}$ which is free and finitely generated both as a left $\overline{k}[t]$-module and a left $\overline{k}[\sigma]$-module and which satisfies, for integers N sufficiently large,

$$(t-\theta)^N\mathcal{M} \subseteq \sigma\mathcal{M}. \tag{3}$$

Given an Anderson t-motive $\mathcal{M}$, let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(\mathcal{M})$ be a $\overline{k}[t]$-basis. Then multiplication by σ on $\mathcal{M}$ is represented by $\sigma\mathbf{m} = \Phi\mathbf{m}$ for some matrix $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$. Note that (3) implies that $\det\Phi = c(t-\theta)^s$ for some $c \in \overline{k}^\times$, $s \geq 0$, and hence $\Phi \in \mathrm{GL}_r(\overline{k}(t))$. We say that $\mathcal{M}$ is rigid analytically trivial if there exists $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that $\Psi^{(-1)} = \Phi\Psi$. We note that Ψ is in fact in $\mathrm{Mat}_r(\mathbb{E})$ by [2, Prop. 3.1.3].

We say that a left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module is a pre-$t$-motive if it is finite dimensional over $\overline{k}(t)$. Let $\mathcal{P}$ be the category of pre-t-motives. Morphisms in $\mathcal{P}$ are left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module homomorphisms, and thus $\mathcal{P}$ forms an abelian category. Given a pre-t-motive P of dimension r over $\overline{k}(t)$, let $\mathbf{p} \in \mathrm{Mat}_{r\times 1}(P)$ be a $\overline{k}(t)$-basis of P. Multiplication by σ on P is given by $\sigma\mathbf{p} = \Phi\mathbf{p}$ for some matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$. We say that P is rigid analytically trivial if there exists $\Psi \in \mathrm{GL}_r(\mathbb{L})$ so that $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization for Φ. It is unique up to right multiplication by a matrix in $\mathrm{GL}_r(\mathbb{F}_q(t))$ [12, §4.1.6].

The Laurent series field $\mathbb{C}_\infty((t))$ carries the natural structure of a left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module by setting $\sigma(f) = f^{(-1)}$. As such, the subfields $\mathbb{L}$ and $\overline{k}(t)$ are $\overline{k}(t)[\sigma,\sigma^{-1}]$-submodules. Similarly $\mathbb{C}_\infty[[t]] \supseteq \mathbb{T} \supseteq \overline{k}[t]$ have natural left $\overline{k}[t,\sigma]$-module structures. For any σ-invariant subring F of $\mathbb{C}_\infty((t))$, we denote by F^σ the subring of elements in F fixed by σ. Then we have

$$\mathbb{T}^\sigma = \overline{k}[t]^\sigma = \mathbb{F}_q[t], \quad \mathbb{L}^\sigma = \overline{k}(t)^\sigma = \mathbb{F}_q(t).$$

See [12, Lem. 3.3.2] for more details.

Now consider $P^\dagger := \mathbb{L} \otimes_{\overline{k}(t)} P$, where we give $P^\dagger$ a left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module structure by letting σ act diagonally:

$$\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m, \quad \forall\, f \in \overline{k}(t), m \in P.$$

Let

$$P^B := (P^\dagger)^\sigma := \{\mu \in P^\dagger : \sigma\mu = \mu\}.$$

Then P^B is a vector space over $\mathbb{F}_q(t)$. The natural map $\mathbb{L} \otimes_{\mathbb{F}_q(t)} P^B \to P^\dagger$ is an isomorphism if and only if P is rigid analytically trivial [12, §3.3]. In this situation, $\Psi^{-1}\mathbf{p}$ is a canonical $\mathbb{F}_q(t)$-basis of P^B, where Ψ is a rigid analytic trivialization for Φ.

Given an Anderson t-motive $\mathcal{M}$, we obtain a pre-t-motive M by setting

$$M := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}$$

and extending the action of σ in the natural way. Thus, $\mathcal{M} \mapsto M$ is a functor from the category of Anderson t-motives to the category of pre-t-motives $\mathcal{P}$. Moreover, one has that the natural map

$$\mathrm{Hom}_{\overline{k}[t,\sigma]}(\mathcal{M},\mathcal{N}) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t) \to \mathrm{Hom}_{\overline{k}(t)[\sigma,\sigma^{-1}]}(M,N) \tag{4}$$

is an isomorphism of $\mathbb{F}_q(t)$-vector spaces for any Anderson t-motives $\mathcal{M}$ and $\mathcal{N}$.

We define the category $\mathcal{A}^I$ of Anderson t-motives up to isogeny to be the category whose objects are Anderson t-motives and whose morphisms, for Anderson t-motives $\mathcal{M}$ and $\mathcal{N}$, are $\mathrm{Hom}_{\mathcal{A}^I}(\mathcal{M},\mathcal{N}) := \mathrm{Hom}_{\overline{k}[t,\sigma]}(\mathcal{M},\mathcal{N}) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t)$. We define the full subcategory $\mathcal{AR}^I$ of rigid analytically trivial Anderson t-motives up to isogeny by restriction. Letting $\mathcal{R}$ be the category of rigid analytically trivial pre-t-motives, $\mathcal{R}$ forms a neutral Tannakian category over $\mathbb{F}_q(t)$ with the fiber functor $P \mapsto P^B$. The functor $\mathcal{M} \mapsto M : \mathcal{AR}^I \to \mathcal{R}$ is fully faithful.

We define the category $\mathcal{T}$ of t-motives to be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the essential image of the functor $\mathcal{M} \mapsto M : \mathcal{AR}^I \to \mathcal{R}$. For any t-motive M, let $\mathcal{T}_M$ be the strictly full Tannakian subcategory of $\mathcal{T}$ generated by M. As $\mathcal{T}_M$ is a neutral Tannakian category over $\mathbb{F}_q(t)$, there is an affine algebraic group scheme Γ_M over $\mathbb{F}_q(t)$ so that $\mathcal{T}_M$ is equivalent to the category of finite dimensional representations of Γ_M over $\mathbb{F}_q(t)$. We call Γ_M the Galois group of M. The main theorem of [12] can be stated as follows.

Theorem 2.2.1 (Papanikolas [12, Thm. 1.1.7])**.** *Let M be a t-motive and let Γ_M be its Galois group. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t)) \cap \mathrm{Mat}_r(\overline{k}[t])$ represents multiplication by σ on M and that $\det\Phi = c(t-\theta)^s$, $c \in \overline{k}^\times$. Let Ψ be a rigid analytic trivialization of Φ in $\mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_r(\mathbb{E})$. Finally, let L be the subfield of $\overline{k_\infty}$ generated by all the entries of $\Psi(\theta)$ over $\overline{k}$. Then*

$$\dim \Gamma_M = \mathrm{tr.deg}_{\overline{k}}\, L.$$

2.3. **Difference Galois groups.** Let M be a t-motive. Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represent multiplication by σ on M and let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ be a rigid analytic trivialization for Φ. One can develop a Galois theory for such systems of difference equations, $\Psi^{(-1)} = \Phi\Psi$, and in turn relate its difference Galois group to Γ_M (see [12, §4]). We describe this construction below.

We define a $\overline{k}(t)$-algebra homomorphism $\nu_\Psi : \overline{k}(t)[X, 1/\det X] \to \mathbb{L}$ by setting $\nu_\Psi(X_{ij}) = \Psi_{ij}$, where $X = (X_{ij})$ is an $r \times r$ matrix of independent variables. We let

$$\mathfrak{p}_\Psi := \mathrm{Ker}\,\nu_\Psi, \quad \Sigma_\Psi := \mathrm{Im}\,\nu_\Psi \subseteq \mathbb{L}, \quad \Lambda_\Psi := \text{fraction field of } \Sigma_\Psi \text{ in } \mathbb{L}.$$

We set $Z_\Psi := \mathrm{Spec}\,\Sigma_\Psi$. In this way, Z_Ψ is the smallest closed subscheme of $\mathrm{GL}_{r/\overline{k}(t)}$ so that $\Psi \in Z_\Psi(\mathbb{L})$.

Now set $\Psi_1, \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L})$ to be the matrices such that $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$, and let $\tilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L})$. We define an $\mathbb{F}_q(t)$-algebra homomorphism $\mu_\Psi : \mathbb{F}_q(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L}$ by setting $\mu(X_{ij}) = \tilde{\Psi}_{ij}$. We let $\Delta_\Psi := \mathrm{Im}\,\mu_\Psi$ and set

$$\Gamma_\Psi := \mathrm{Spec}\,\Delta_\Psi. \tag{5}$$

Thus Γ_Ψ is the smallest closed subscheme of $\mathrm{GL}_{r/\mathbb{F}_q(t)}$ such that $\tilde{\Psi} \in \Gamma_\Psi(\mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L})$.

Theorem 2.3.1 (Papanikolas [12, §4])**.** *Let M be a t-motive. Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represent multiplication by σ on M and let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfy $\Psi^{(-1)} = \Phi\Psi$.*

(a) *Γ_Ψ is a closed $\mathbb{F}_q(t)$-subgroup scheme of $\mathrm{GL}_{r/\mathbb{F}_q(t)}$.*

(b) *Γ_Ψ is absolutely irreducible and smooth over $\overline{\mathbb{F}_q(t)}$.*

(c) *Z_Ψ is stable under right-multiplication by $\Gamma_\Psi \times_{\mathbb{F}_q(t)} \overline{k}(t)$ and is a torsor for $\Gamma_\Psi \times_{\mathbb{F}_q(t)} \overline{k}(t)$ over $\overline{k}(t)$.*
(d) $\dim \Gamma_\Psi = \operatorname{tr.deg}_{\overline{k}(t)} \Lambda_\Psi$.
(e) *Γ_Ψ is isomorphic to Γ_M over $\mathbb{F}_q(t)$.*

Remark 2.3.2. Theorem 2.3.1 implies that Γ_Ψ can be regarded as a linear algebraic group over $\mathbb{F}_q(t)$.

2.4. **Rank 2 Drinfeld modules and t-motives.** Let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined over $\overline{k}$ given by $\rho_t = \theta + \kappa\tau + u\tau^2$, κ, $u \in \overline{k}$. Put

$$\Phi_\rho := \begin{bmatrix} 0 & 1 \\ (t-\theta)/u^{(-2)} & -\kappa^{(-1)}/u^{(-2)} \end{bmatrix} \in \operatorname{Mat}_2(\overline{k}[t]).$$

Let $\mathcal{M}_\rho$ be a left $\overline{k}[t]$-module which is free of rank 2 over $\overline{k}[t]$ with a basis $\mathbf{m} = [m_1, m_2]^{\mathrm{tr}} \in \operatorname{Mat}_{2\times 1}(\mathcal{M}_\rho)$. We give $\mathcal{M}_\rho$ the structure of a left $\overline{k}[t,\sigma]$-module by defining $\sigma\mathbf{m} = \Phi_\rho\mathbf{m}$.

Lemma 2.4.1. *The left $\overline{k}[t,\sigma]$-module $\mathcal{M}_\rho$ is an Anderson t-motive.*

Proof. We claim that $\mathcal{M}_\rho = \overline{k}[\sigma]m_1$, whence $\mathcal{M}_\rho$ is free over $\overline{k}[\sigma]$ since $\sigma : \mathcal{M}_\rho \to \mathcal{M}_\rho$ is injective. Note that since $m_2 = \sigma m_1$, we have $m_2 \in \overline{k}[\sigma]m_1$. Thus, $tm_1 \in \overline{k}[\sigma]m_1$ also. Using that $\sigma t m_1 = t\sigma m_1 \in \overline{k}[\sigma]m_1$, we have that tm_2 lies in $\overline{k}[\sigma]m_1$. Similarly, $\sigma t m_2 = t\sigma m_2 \in \overline{k}[\sigma]m_1$ shows that $t^2 m_1 \in \overline{k}[\sigma]m_1$. Continuing this argument we see that $t^n m_1, t^n m_2 \in \overline{k}[\sigma]m_1$ for all $n \geq 0$. This proves our claim. On the other hand, we find easily that $(t-\theta)\mathcal{M}_\rho \subseteq \sigma\mathcal{M}_\rho$, and hence $\mathcal{M}_\rho$ is an Anderson t-motive. $\square$

Following unpublished work of Anderson, we will show that the assignment $\rho \mapsto \mathcal{M}_\rho$ is a covariant functor from the category of Drinfeld modules of rank 2 over $\overline{k}$ to the category of Anderson t-motives. First we recall Ore's τ-adjoint operation [10, §1.7],

$$f \mapsto f^* : \overline{k}[\tau] \to \overline{k}[\sigma],$$

where if $f = \sum a_i \tau^i$, then

$$f^* := \sum a_i^{(-i)} \sigma^i.$$

One has that $(fg)^* = g^* f^*$ for all f, $g \in \overline{k}[\tau]$, and moreover $f \mapsto f^*$ defines an anti-isomorphism of rings $\overline{k}[\tau] \to \overline{k}[\sigma]$.

Suppose ρ, $\rho' : \mathbb{F}_q[t] \to \overline{k}[t]$ are Drinfeld modules of rank 2 over $\overline{k}$, and assign κ', u', $\{m_1', m_2'\}$, Φ', and $\mathcal{M}_{\rho'}$ as in the beginning of the section. Now $e \in \overline{k}[\tau]$ defines a morphism $e : \rho \to \rho'$ if

$$e\rho_a = \rho'_a e, \quad \forall\, a \in \mathbb{F}_q[t].$$

As we observed in the proof of Lemma 2.4.1, $\mathcal{M}_\rho = \overline{k}[\sigma]m_1$ and $\mathcal{M}_{\rho'} = \overline{k}[\sigma]m_1'$. Define a $\overline{k}[\sigma]$-linear function

$$\varepsilon : \mathcal{M}_\rho \to \mathcal{M}_{\rho'}$$

such that $\varepsilon(m_1) = e^* m_1'$.

Lemma 2.4.2. *The map $\varepsilon : \mathcal{M}_\rho \to \mathcal{M}_{\rho'}$ is a morphism of Anderson t-motives.*

Proof. By definition ε is $\overline{k}[\sigma]$-linear, so it suffices to show that it commutes with t. We check easily that

$$tm_1 = (\rho_t)^* m_1, \quad tm_1' = (\rho_t')^* m_1'. \tag{6}$$

Therefore,

$$\varepsilon(tm_1) = (\rho_t)^* \varepsilon(m_1) = (\rho_t)^* e^* m_1' = e^*(\rho_t')^* m_1' = e^* t m_1' = t e^* m_1' = t\varepsilon(m_1),$$

and so ε is $\overline{k}[t]$-linear. □

It is simple to check that the construction of ε behaves well under composition of morphisms, and thus we have defined a functor $\rho \mapsto \mathcal{M}_\rho$ as desired.

Proposition 2.4.3. *The functor $\rho \mapsto \mathcal{M}_\rho$ from rank 2 Drinfeld modules over $\overline{k}$ to the category of Anderson t-motives is fully faithful.*

Proof. We continue with the notation as above for two rank 2 Drinfeld modules ρ, ρ'. Certainly $\varepsilon = 0$ if and only if $e = 0$, and so faithfulness is immediate. Now suppose $h : \mathcal{M}_\rho \to \mathcal{M}_{\rho'}$ is a morphism. Then for some $e \in \overline{k}[\tau]$, $h(m_1) = e^* m_1'$. Immediately from (6) we see that

$$(\rho_t)^* e^* m_1' = h((\rho_t)^* m_1) = h(tm_1) = th(m_1) = te^* m_1' = e^*(\rho_t')^* m_1'.$$

Thus $e\rho_t = \rho_t' e$, and $e : \rho \to \rho'$ is a morphism of Drinfeld modules. Moreover, h is the morphism $\mathcal{M}_\rho \to \mathcal{M}_{\rho'}$ associated to e. □

Remark 2.4.4. By Proposition 2.4.3, we see for a rank 2 Drinfeld module ρ that the ring $\mathrm{End}(\rho)$ is isomorphic to $\mathrm{End}_{\overline{k}[t,\sigma]}(\mathcal{M}_\rho)$. In particular, when ρ does not have complex multiplication, by (4) we have $\mathrm{End}_{\overline{k}(t)[\sigma,\sigma^{-1}]}(M_\rho) = \mathbb{F}_q(t)$, where $M_\rho := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}_\rho$.

2.5. **Rigid analytic trivializations of rank 2 Drinfeld modules.** We now review how to create a rigid analytic trivialization $\Psi_\rho \in \mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_2(\mathbb{E})$ for Φ_ρ and connect it with the period matrix of ρ (for more details, see [13, §4.2]). For simplicity we assume that $\rho : \mathbb{F}_q[t] \to \overline{k}[\tau]$ satisfies

$$\rho_t = \theta + \kappa\tau + \tau^2, \quad \kappa \in \overline{k}.$$

For applications we do not lose any generality (see Remark 3.4.2).

Let $\exp_\rho(z) := z + \sum_{i=1}^{\infty} \alpha_i z^{q^i}$ be the exponential function of ρ. Given $u \in \mathbb{C}_\infty$ we consider the Anderson generating function

$$f_u(t) := \sum_{i=0}^{\infty} \exp_\rho\left(\frac{u}{\theta^{i+1}}\right) t^i = \sum_{i=0}^{\infty} \frac{\alpha_i u^{q^i}}{\theta^{q^i} - t} \in \mathbb{T} \tag{7}$$

and note that $f_u(t)$ is a meromorphic function on $\mathbb{C}_\infty$. It has simple poles at $\theta, \theta^q, \ldots$ with residues $-u, -\alpha_1 u^q, \ldots$ respectively. Using that $\rho_t(\exp_\rho(u/\theta^{i+1})) = \exp_\rho(u/\theta^i)$, we have

$$\kappa f_u^{(1)}(t) + f_u^{(2)} = (t-\theta) f_u(t) + \exp_\rho(u). \tag{8}$$

Since $f_u^{(m)}(t)$ converges away from $\{\theta^{q^m}, \theta^{q^{m+1}}, \ldots\}$ and $\mathrm{Res}_{t=\theta} f_u(t) = -u$, we have

$$\kappa f_u^{(1)}(\theta) + f_u^{(2)}(\theta) = -u + \exp_\rho(u) \tag{9}$$

by specializing (8) at $t = \theta$.

Fixing an $\mathbb{F}_q[\theta]$-basis $\{\omega_1, \omega_2\}$ of $\Lambda_\rho := \mathrm{Ker}\exp_\rho$, we set $f_i := f_{\omega_i}(t)$ for $i = 1, 2$. Recall the analogue of the Legendre relation proved by Anderson,

$$\omega_1 F_\tau(\omega_2) - \omega_2 F_\tau(\omega_1) = \widetilde{\pi}/\xi, \tag{10}$$

where $\xi \in \overline{\mathbb{F}_q}^{\times}$ satisfies $\xi^{(-1)} = -\xi$ and $\widetilde{\pi}$ is a generator of the period lattice of the Carlitz module C. We pick a suitable $(q-1)$-st root of $-\theta$ so that $\Omega(\theta) = -1/\widetilde{\pi}$, where

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{E},$$

(see [2, Cor. 5.1.4]). Now put

$$\Psi_\rho := \xi\Omega \begin{bmatrix} -f_2^{(1)} & f_1^{(1)} \\ \kappa f_2^{(1)} + f_2^{(2)} & -\kappa f_1^{(1)} - f_1^{(2)} \end{bmatrix}.$$

By (8) we have $\Psi_\rho^{(-1)} = \Phi_\rho \Psi_\rho$, and thus $\det(\Psi_\rho)$ is an $\mathbb{F}_q(t)^\times$-multiple of $\xi\Omega$. By specializing at $t = \theta$ and using the Legendre relation, we see that $\det(\Psi_\rho) = \xi\Omega$. Therefore, $\Psi_\rho \in \mathrm{GL}_2(\mathbb{T})$, and $\mathcal{M}_\rho$ is rigid analytically trivial. Hence $M_\rho := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}_\rho$ is a t-motive. Since

$$F_\tau(\omega_i) = \sum_{j=0}^{\infty} \exp_\rho\left(\frac{\omega_i}{\theta^{j+1}}\right)^q \theta^j, \quad i = 1, 2,$$

(cf. [15, §6.4]), by evaluating Ψ_ρ at $t = \theta$ we obtain

$$\Psi_\rho(\theta) = \frac{\xi}{\widetilde{\pi}} \begin{bmatrix} F_\tau(\omega_2) & -F_\tau(\omega_1) \\ \omega_2 & -\omega_1 \end{bmatrix}. \tag{11}$$

Hence $\Psi_\rho^{-1}(\theta) = P_\rho$ (see (2)) and $\overline{k}(\Psi_\rho(\theta)) = \overline{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2))$. Therefore by Theorem 2.2.1 we have proved the following equivalence.

Proposition 2.5.1. *Let ρ be a rank 2 Drinfeld module defined by $\rho_t = \theta + \kappa\tau + \tau^2$ with $\kappa \in \overline{k}$. Let ω_1, ω_2 generate the period lattice of ρ. Then*

$$\Gamma_{M_\rho} = \mathrm{GL}_2 \Leftrightarrow \operatorname{tr.deg}_{\overline{k}} \overline{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)) = 4.$$

3. Algebraic independence of periods and quasi-periods

3.1. **The structure of the motivic Galois group Γ_M.** From now on, we fix a rank 2 Drinfeld $\mathbb{F}_q[t]$-module ρ over $\overline{k}$, given by $\rho_t := \theta + \kappa\tau + \tau^2$, $\kappa \in \overline{k}$. As in §2.4, we put

$$\Phi := \Phi_\rho := \begin{bmatrix} 0 & 1 \\ (t-\theta) & -\kappa^{(-1)} \end{bmatrix},$$

and let $M := M_\rho$ be its associated t-motive, with $\overline{k}(t)$-basis $\mathbf{m} := [m_1, m_2]^{\mathrm{tr}} \in \mathrm{Mat}_{2\times 1}(M)$.

Lemma 3.1.1. *Let ρ and $(M, \mathbf{m}, \Phi)$ be defined as above. Then M is simple as a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module.*

Proof. Let N be a non-zero proper left $\overline{k}(t)[\sigma, \sigma^{-1}]$-submodule of M. Since N is invariant under the σ-action, it is spanned over $\overline{k}(t)$ by $fm_1 + m_2$ for some $f \in \overline{k}(t)^\times$. Because

$$\sigma(fm_1 + m_2) = \beta(fm_1 + m_2)$$

for some $\beta \in \overline{k}(t)^\times$, we have the equalities in $\overline{k}(t)$,

$$f^{(-1)} - \kappa^{(-1)} = \beta, \quad (t - \theta) = \beta f.$$

If $\deg_t(f) > 0$, the first equality implies that $\deg_t f = \deg_t \beta$, and therefore by the second equality we have $1 = 2 \cdot \deg_t f$, which is a contradiction. If $\deg_t(f) \leq 0$, then $\deg_t(\beta) \leq 0$, which also contradicts the second equality. □

We now describe the tautological representation of the Galois group Γ_M. Since $\mathcal{T}_M$ is a neutral Tannakian category over $\mathbb{F}_q(t)$, we have a canonically defined faithful representation

$$\varphi : \Gamma_M \hookrightarrow \mathrm{GL}(M^B).$$

Throughout this paper, we always identify Γ_Ψ with Γ_M (cf. Theorem 2.3.1). The entries of $\Psi^{-1}\mathbf{m}$ form a canonical $\mathbb{F}_q(t)$-basis of M^B (see [12, Prop. 3.3.8(b)]), and so by [12, Thm. 4.5.3], the representation φ can be described as follows: for any $\mathbb{F}_q(t)$-algebra R,

$$\begin{aligned} \varphi : \Gamma_\Psi(R) &\hookrightarrow \mathrm{GL}(R \otimes_{\mathbb{F}_q(t)} M^B) \\ \gamma &\mapsto \big((1 \otimes \Psi^{-1}\mathbf{m}) \mapsto (\gamma^{-1} \otimes 1)(1 \otimes \Psi^{-1}\mathbf{m})\big), \end{aligned} \tag{12}$$

The representation φ is called the tautological representation of Γ_M.

Proposition 3.1.2. *Let ρ and $(M, \mathbf{m}, \Phi)$ be defined as above. Then the determinant map* $\det : \Gamma_M \to \mathbb{G}_m$ *is surjective.*

Proof. We consider the tensor product of $M^{\otimes 2} := M \otimes_{\overline{k}(t)} M$, on which σ acts diagonally. Note that $\mathbf{m} \otimes \mathbf{m}$ defines a canonical $\overline{k}(t)$-basis of $M^{\otimes 2}$ and that the Kronecker product matrix $\Phi \otimes \Phi$ represents multiplication by σ on $M^{\otimes 2}$ with respect to $\mathbf{m} \otimes \mathbf{m}$. Furthermore, $\Psi \otimes \Psi$ provides a rigid analytic trivialization of $\Phi \otimes \Phi$.

Let $N := \bigwedge^2 M$ be the space $\overline{k}(t) \cdot \mathbf{n}$, where $\mathbf{n} := m_1 \otimes m_2 - m_2 \otimes m_1$. By direct computation, $\sigma \mathbf{n} = \det \Phi \cdot \mathbf{n}$. Hence N is a sub-t-motive of $M^{\otimes 2}$, since

$$(\det \Psi)^{(-1)} = (\det \Phi)(\det \Psi).$$

One checks that $\frac{\det \Psi}{\xi \Omega}$ is fixed by σ, and hence $\frac{\det \Psi}{\xi \Omega} \in \mathbb{F}_q(t)^\times$. Since Ω has infinitely many zeros, $\det \Psi$ is transcendental over $\overline{k}(t)$. Thus, by Theorem 2.3.1 we see that the algebraic group Γ_N is isomorphic to $\mathbb{G}_m$. Since N is a sub-t-motive of $M^{\otimes 2}$, we have a surjection $\Gamma_{M^{\otimes 2}} \twoheadrightarrow \Gamma_N \cong \mathbb{G}_m$. As $M^{\otimes 2}$ is an object in the Tannakian category $\mathcal{T}_M$, we also have a surjective map $\Gamma_M \twoheadrightarrow \Gamma_{M^{\otimes 2}}$. Composing these two surjective maps, we have a surjection

$$\Gamma_M \twoheadrightarrow \mathbb{G}_m.$$

Let $\mathcal{T}_{M^{\otimes 2}}$ and $\mathcal{T}_N$ be the Tannakian subcategories of $\mathcal{T}_M$ generated by $M^{\otimes 2}$ and N respectively. Note that the fiber functor of $\mathcal{T}_{M^{\otimes 2}}$ is the restriction of the fiber functor of $\mathcal{T}_M$ to $\mathcal{T}_{M^{\otimes 2}}$, and the fiber functor of $\mathcal{T}_N$ is the restriction of the fiber functor of $\mathcal{T}_{M^{\otimes 2}}$ to $\mathcal{T}_N$. Using this property and (12), we see that, for any $\mathbb{F}_q(t)$-algebra R, the restriction of the action of any $\gamma \in \Gamma_M(R)$ to the R-basis $1 \otimes (\det \Psi)^{-1}\mathbf{n}$ of $R \otimes_{\mathbb{F}_q(t)} N^B$ is equal to the action of $\det \gamma$. Hence, the composition map $\Gamma_M \twoheadrightarrow \mathbb{G}_m$ is equal to the determinant map. $\square$

Corollary 3.1.3. *Let ρ and $(M, \mathbf{m}, \Phi)$ be as above. If $\dim \Gamma_M \leq 3$, then Γ_M is solvable.*

Proof. By Proposition 3.1.2, we consider the following short exact sequence of linear algebraic groups

$$1 \to K \to \Gamma_M \overset{\det}{\twoheadrightarrow} \mathbb{G}_m \to 1.$$

Let K^0 be the identity component of K. Since K is normal in Γ_M, for any $g \in \Gamma_M$ we have $g^{-1}Kg = K$ and hence $g^{-1}K^0 g = K^0$. We note that K^0 is solvable since $\dim K^0 \leq 2$ (see [11, p. 137]), and Γ_M/K^0 is abelian since it is a one-dimensional connected algebraic group (see [11, p. 126]). It follows that Γ_M is solvable. $\square$

3.2. **An analogue of the motivic version of Tate's conjecture.** We let $\mathrm{End}(M)$ denote the endomorphisms of M as a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module. Given $f \in \mathrm{End}(M)$, we have $f(\mathbf{m}) = F\mathbf{m}$ for some $F \in \mathrm{Mat}_2(\overline{k}(t))$. Since $f\sigma = \sigma f$, we have

$$\Phi F = F^{(-1)}\Phi.$$

From this equation we see that the matrix $\Psi^{-1}F\Psi \in \mathrm{Mat}_2(\mathbb{L})$ is fixed by σ, and hence $\Psi^{-1}F\Psi \in \mathrm{Mat}_2(\mathbb{F}_q(t))$. Thus, we have the following injective map:

$$\begin{aligned} \mathrm{End}(M) &\hookrightarrow \mathrm{End}(M^B) = \mathrm{Mat}_2(\mathbb{F}_q(t)), \\ f &\mapsto f^B := \Psi^{-1}F\Psi. \end{aligned}$$

Since the representation $\varphi : \Gamma_\Psi \to \mathrm{GL}(M^B)$ is functorial in M (cf. [12, Thm. 4.5.3]), it follows that for any $\gamma \in \Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ we have the commutative diagram

$$\begin{array}{ccc} \overline{\mathbb{F}_q(t)} \otimes_{\mathbb{F}_q(t)} M^B & \xrightarrow{\varphi(\gamma)} & \overline{\mathbb{F}_q(t)} \otimes_{\mathbb{F}_q(t)} M^B \\ \Big\downarrow {\scriptstyle 1\otimes f^B} & & \Big\downarrow {\scriptstyle 1\otimes f^B} \\ \overline{\mathbb{F}_q(t)} \otimes_{\mathbb{F}_q(t)} M^B & \xrightarrow{\varphi(\gamma)} & \overline{\mathbb{F}_q(t)} \otimes_{\mathbb{F}_q(t)} M^B. \end{array}$$

Thus, we have defined an injective map

$$\begin{aligned} \mathrm{End}(M) &\hookrightarrow \mathrm{Cent}_{\mathrm{Mat}_2(\mathbb{F}_q(t))}(\Gamma_\Psi(\overline{\mathbb{F}_q(t)})), \\ f &\mapsto f^B := \Psi^{-1}F\Psi. \end{aligned}$$

Theorem 3.2.1. *The natural map*

$$f \mapsto f^B : \mathrm{End}(M) \to \mathrm{Cent}_{\mathrm{Mat}_2(\mathbb{F}_q(t))}(\Gamma_\Psi(\overline{\mathbb{F}_q(t)})),$$

as defined above, is an isomorphism.

Proof. Given $D \in \mathrm{Cent}_{\mathrm{Mat}_2(\mathbb{F}_q(t))}(\Gamma_\Psi(\overline{\mathbb{F}_q(t)}))$, we set $F := \Psi D \Psi^{-1} \in \mathrm{Mat}_2(\Lambda_\Psi)$. We claim that F is in fact in $\mathrm{Mat}_2(\overline{k}(t))$. By [12, Thm. 4.4.6(b)], we need only show that the entries of F are fixed by the action of every $\gamma \in \Gamma_\Psi(\overline{\mathbb{F}_q(t)})$.

For any $\gamma \in \Gamma_\Psi(\overline{\mathbb{F}_q(t)})$, the action of γ on $h(\Psi)$, $h \in \overline{k(t)}[X, 1/\det X]$, is given by

$$\gamma * h(\Psi) := h(\Psi\gamma)$$

(cf. [12, §4.4.3, §4.4.4]). Thus the action of γ on entries of F is given as follows:

$$\gamma * F_{ij} = \big((\Psi\gamma)D(\Psi\gamma)^{-1}\big)_{ij} = (\Psi D \Psi^{-1})_{ij} = F_{ij},$$

since by definition D commutes with γ. Thus $F \in \mathrm{Mat}_2(\overline{k}(t))$, and therefore F induces a $\overline{k}(t)$-linear map on M.

To show that $F \in \mathrm{End}(M)$, it is equivalent to show that $F^{(-1)}\Phi = \Phi F$. The calculation,

$$F^{(-1)}\Phi = \Phi\Psi D\Psi^{-1}\Phi^{-1}\Phi = \Phi F,$$

then completes the proof. □

3.3. **Motivic Galois representations.** In odd characteristic we see in the following theorem that the tautological representation φ is semisimple.

Theorem 3.3.1. *Suppose that p is odd. Let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module over $\overline{k}$ with*

$$\rho_t := \theta + \kappa\tau + \tau^2, \quad \kappa \in \overline{k}.$$

Let M be the t-motive associated to ρ (as in §3.1). Then the faithful representation

$$\varphi : \Gamma_M \hookrightarrow \mathrm{GL}\big(M^B\big)$$

is semisimple.

To prove Theorem 3.3.1, we need the following reduction Lemma.

Lemma 3.3.2. *Let $\mathbb{F}$ be the separable closure of $\mathbb{F}_q(t)$ in $\overline{\mathbb{F}_q(t)}$. Then the group representation $\varphi_{\mathbb{F}} : \Gamma_M(\mathbb{F}) \hookrightarrow \mathrm{GL}(\mathbb{F} \otimes_{\mathbb{F}_q(t)} M^B)$ is semisimple.*

Proof. Let $(\mathbf{m}, \Phi, \Psi)$ be defined as in §3.1. Suppose that there exists a one-dimensional $\mathbb{F}$-vector space $V \subseteq \mathbb{F} \otimes_{\mathbb{F}_q(t)} M^B$, which is invariant under $\Gamma_\Psi(\mathbb{F})$. Let $[u, v]\Psi^{-1}\mathbf{m}$, for u, $v \in \mathbb{F}$, be an $\mathbb{F}$-basis of V.

Step 1: We claim that $u \neq 0$ and $v \neq 0$. On the contrary, suppose that $u = 0$ or $v = 0$. Without loss of generality, we may assume that $u = 1$, $v = 0$. Let $\mathbb{B}$ be the Borel group consisting of all lower triangular matrices in GL_2. Since V is invariant under $\Gamma_\Psi(\mathbb{F})$, we see that $\Gamma_\Psi(\mathbb{F}) \subseteq \mathbb{B}(\mathbb{F})$. Moreover, since $\mathbb{B}$ is a closed subgroup of GL_2 and $\Gamma_\Psi(\mathbb{F})$ is dense in Γ_Ψ (see [14, Lem. 11.2.5]), we see that

$$\Gamma_\Psi \subseteq \mathbb{B}.$$

Thus $[1, 0]\Psi^{-1}\mathbf{m} \in M^B$ generates a sub-representation of φ. This sub-representation corresponds to a nontrivial proper sub-t-motive of M by [12, Prop. 4.5.8], which contradicts Lemma 3.1.1. Thus we may assume that V is spanned by

$$[e, 1]\Psi^{-1}\mathbf{m}, \quad e \in \mathbb{F}^\times.$$

Step 2: We claim that $e \in \mathbb{F}^\times \setminus \mathbb{F}_q(t)^\times$. If $e \in \mathbb{F}_q(t)^\times$, then having $[e, 1]\Psi^{-1}\mathbf{m} \in M^B$ implies that there is a conjugation embedding over $\mathbb{F}_q(t)$,

$$\Gamma_\Psi(\mathbb{F}) \hookrightarrow \mathbb{B}(\mathbb{F}).$$

Taking the Zariski closure of $\Gamma_\Psi(\mathbb{F})$ inside GL_2, we see that Γ_Ψ is embedded via conjugation into $\mathbb{B}$ over $\mathbb{F}_q(t)$. This provides a sub-representation of φ. Using the argument in Step 1, we obtain a contradiction. Thus $e \in \mathbb{F}^\times \setminus \mathbb{F}_q(t)^\times$.

Now since $e \in \mathbb{F}^\times \setminus \mathbb{F}_q(t)^\times$, we can choose an automorphism η of the field $\mathbb{F}$ over $\mathbb{F}_q(t)$ so that

$$\eta(e) \neq e.$$

Step 3: We claim that $[\eta(e), 1]\Psi^{-1}\mathbf{m}$ spans an $\mathbb{F}$-invariant subspace of $\Gamma_\Psi(\mathbb{F})$. Since $[e, 1]\Psi^{-1}\mathbf{m}$ spans an invariant subspace of $\Gamma_\Psi(\mathbb{F})$, for any $\gamma \in \Gamma_\Psi(\mathbb{F})$ we have

$$\varphi(\gamma)[e, 1]\Psi^{-1}\mathbf{m} := [e, 1]\gamma^{-1}\Psi^{-1}\mathbf{m} = \beta_\gamma[e, 1]\Psi^{-1}\mathbf{m},$$

for some $\beta_\gamma \in \mathbb{F}^\times$. Thus we have

$$[e, 1]\gamma^{-1} = \beta_\gamma[e, 1], \quad \text{for all } \gamma \in \Gamma_\Psi(\mathbb{F}).$$

Since Γ_Ψ is defined over $\mathbb{F}_q(t)$, the action of η on both sides of the above equation implies that $[\eta(e), 1]\Psi^{-1}\mathbf{m}$ is a common eigenvector for all $\gamma \in \Gamma_\Psi(\mathbb{F})$, which proves the claim.

Since $\eta(e) \neq e$, $[e, 1]\Psi^{-1}\mathbf{m}$ and $[\eta(e), 1]\Psi^{-1}\mathbf{m}$ are linearly independent over $\mathbb{F}$, and hence the group representation $\varphi_{\mathbb{F}}$ is semisimple. □

Proof of Theorem 3.3.1. Suppose there exist u, $v \in \overline{\mathbb{F}_q(t)}$ so that $[u,v]\Psi^{-1}\mathbf{m}$ spans a one-dimensional $\overline{\mathbb{F}_q(t)}$-vector space V that is invariant under $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$. Using the argument in Step 1 of the proof of the lemma, we see that $uv \neq 0$ and hence V is spanned by $[e,1]\Psi^{-1}\mathbf{m}$, $e := u/v$. Using the argument in Step 2 of the lemma, we see that

$$e \in \overline{\mathbb{F}_q(t)}^\times \setminus \mathbb{F}_q(t)^\times.$$

We claim that the separable degree of e over $\mathbb{F}_q(t)$ is strictly larger than 1. Then the proof of Theorem 3.3.1 will be completed by using the argument in Step 3 of the lemma.

Since $[e,1]\Psi^{-1}\mathbf{m}$ is a common eigenvector for $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$, we find in particular that for any

$$\gamma = \begin{bmatrix} x & y \\ z & w \end{bmatrix}^{-1} \in \Gamma_\Psi(\mathbb{F}),$$

we have $[e,1]\gamma^{-1}\Psi^{-1}\mathbf{m} = \beta_\gamma[e,1]\Psi^{-1}\mathbf{m}$, for some $\beta_\gamma \in \overline{\mathbb{F}_q(t)}^\times$. Thus, we have the equality

$$[e,1]\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \beta_\gamma[e,1]$$

which induces the quadratic relation

$$ye^2 + (w-x)e - z = 0.$$

If there exists some $\gamma = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]^{-1} \in \Gamma_\Psi(\mathbb{F})$ with $y \neq 0$, then e must be separable over $\mathbb{F}_q(t)$ since the characteristic p is odd and x, y, z, $w \in \mathbb{F}$. Hence in that case the claim above is proved. Thus we need only consider the other case that $\Gamma_\Psi(\mathbb{F}) \subseteq \mathbb{B}$. But in this case, the argument of Step 1 above gives a contradiction, and hence the proof is completed. □

Remark 3.3.3. We briefly discuss what happens when $p = 2$. Let $q = 2^\ell$ for some $\ell \geq 1$. First, let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined over $\bar{k}$ with complex multiplication, and let $K = \mathrm{End}(M_\rho)$. In general, one finds directly that Γ_{M_ρ} is isomorphic to the restriction of scalars of $\mathbb{G}_m$ from K to $\mathbb{F}_q(t)$, and so its tautological representation is semisimple as long as $K/\mathbb{F}_q(t)$ is separable (see [6, Prop. 3.3.1] for a similar calculation). However, consider the Drinfeld module ρ' defined by $\rho'_t = \theta + (\sqrt{\theta} + \sqrt{\theta^q})\tau + \tau^2$. In this case $\mathrm{End}(M_{\rho'})$ is $\mathbb{F}_q(\sqrt{t})$, which is inseparable over $\mathbb{F}_q(t)$. In particular, one finds

$$\Gamma_{M_\rho}(\overline{\mathbb{F}_q(t)}) = \left\{ \begin{bmatrix} a & b \\ tb & a \end{bmatrix} \,\middle|\, a, b \in \overline{\mathbb{F}_q(t)}, a^2 + tb^2 \neq 0 \right\},$$

and so the tautological representation of $\Gamma_{M_{\rho'}}$ is not semisimple, whence the conclusion of Theorem 3.3.1 is false for ρ'. Now every rank 2 Drinfeld $\mathbb{F}_q[t]$-module with complex multiplication whose endomorphism algebra is inseparable over $\mathbb{F}_q(t)$ is isomorphic to ρ', so it is essentially the only counterexample when there are extra endomorphisms.

However, for a rank 2 Drinfeld $\mathbb{F}_q[t]$-module without complex multiplication, it is not clear to the authors if the proof of Theorem 3.3.1 provides enough information to decide whether the tautological representation of the Galois group is semisimple. We expect nevertheless in this case that the Galois group is $\mathrm{GL}_{2/\mathbb{F}_q(t)}$.

3.4. **Algebraic independence of periods and quasi-periods.** We continue with the notation of the previous sections, but from now on we assume that ρ does not have complex multiplication. That is, we assume $\mathrm{End}(M) = \mathbb{F}_q(t)$ (cf. Remark 2.4.4). Our main goal of this section is to prove the following theorem.

Theorem 3.4.1. *Suppose that p is odd. Let ρ be a rank 2 Drinfeld $\mathbb{F}_q[t]$-module over $\overline{k}$ with*

$$\rho_t := \theta + \kappa\tau + \tau^2, \quad \kappa \in \overline{k},$$

without complex multiplication. Let M be the t-motive associated to ρ, as in §3.1. Then

$$\Gamma_M = \mathrm{GL}_2 .$$

In particular, the 4 quantities,

$$\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2),$$

are algebraically independent over $\overline{k}$.

Remark 3.4.2. The above theorem still holds for an arbitrary rank 2 Drinfeld module ρ over $\overline{k}$ without complex multiplication, still under the assumption $p \neq 2$. This can be explained as follows. Let $u \in \overline{k}^\times$ be the coefficient of τ^2 in ρ_t. We pick $x \in \overline{k}^\times$ so that $x^{q^2-1} = 1/u$, and by a simple change of variables we define ν to be the Drinfeld $\mathbb{F}_q[t]$-module over $\overline{k}$ given by $\nu_t := x^{-1}\rho_t x$, which is isomorphic to ρ over $\overline{k}$ (see [10, 15]). Note that the coefficient of τ^2 in ν_t is 1. We let $\mathcal{F}_\tau$ be the quasi-periodic function of ν associated to the biderivation defined by $t \mapsto \tau$. One checks that

$$F_\tau(z) = x^q \mathcal{F}_\tau(x^{-1}z).$$

More generally, given $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ such that $\exp_\rho(\lambda_i) \in \overline{k}$ for $i = 1, \ldots, m$, we note that we have $\exp_\nu(x^{-1}\lambda_i) \in \overline{k}$ for each i. Moreover, we see that

$$\overline{k}\big(\cup_{i=1}^m \{\lambda_i, F_\tau(\lambda_i)\}\big) = \overline{k}\big(\cup_{i=1}^m \{x^{-1}\lambda_i, \mathcal{F}_\tau(x^{-1}\lambda_i)\}\big).$$

This property will be used in the proof of the main theorem of the last section.

The proof of Theorem 3.4.1 relies on the following lemma that shows that, when ρ does not have complex multiplication, the representation φ is absolutely irreducible.

Lemma 3.4.3. *Continuing with the notation of Theorem 3.4.1, the representation $\varphi : \Gamma_M \hookrightarrow \mathrm{GL}(M^B)$ is absolutely irreducible.*

Proof. Suppose that φ is not absolutely irreducible. Then by Theorem 3.3.1, we have an embedding by conjugation over $\overline{\mathbb{F}_q(t)}$,

$$\Gamma_\Psi \hookrightarrow \left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \in \mathrm{GL}_2 \right\},$$

and hence Γ_Ψ is a (non-split) torus over $\mathbb{F}_q(t)$. Since Γ_Ψ is a torus over $\mathbb{F}_q(t)$, we see that

$$\Gamma_\Psi(\mathbb{F}_q(t)) \subseteq \mathbf{C} := \mathrm{Cent}_{\mathrm{Mat}_2(\mathbb{F}_q(t))}\big(\Gamma_\Psi(\overline{\mathbb{F}_q(t)})\big).$$

Suppose there exists $\gamma \in \Gamma_\Psi(\mathbb{F}_q(t))$ so that γ is not an $\mathbb{F}_q(t)^\times$-scalar multiple of the identity matrix Id_2. Then Id_2, $\gamma \in \mathbf{C}$ are linearly independent over $\mathbb{F}_q(t)$. Hence, $\dim_{\mathbb{F}_q(t)} \mathbf{C} \geq 2$, which contradicts Theorem 3.2.1.

Thus, $\Gamma_\Psi(\mathbb{F}_q(t))$ is contained in the one-dimensional torus $\mathbb{G}$ consisting of all $a \cdot \mathrm{Id}_2$, $a \in \overline{\mathbb{F}_q(t)}^\times$. Note that, by [14, Lem. 13.2.7(ii)], $\Gamma_\Psi(\mathbb{F}_q(t))$ is dense in Γ_Ψ. Taking the Zariski closure of $\Gamma_\Psi(\mathbb{F}_q(t))$ inside GL_2, we see that $\Gamma_\Psi \subseteq \mathbb{G}$ and hence M^B splits. The splitting of M^B implies that M is not simple by [12, Prop. 4.5.8], which contradicts Lemma 3.1.1. □

Proof of Theorem 3.4.1. Suppose $\Gamma_M \subsetneq \mathrm{GL}_2$. Then $\dim \Gamma_M \leq 3$, since Γ_M is connected. By Corollary 3.1.3, we see that Γ_M is solvable, which contradicts the absolute irreducibility of the representation φ from the lemma. Thus, $\Gamma_M = \mathrm{GL}_2$. Moreover, the algebraic independence of ω_1, ω_2, $F_\tau(\omega_1)$, $F_\tau(\omega_2)$ over $\overline{k}$ follows from Proposition 2.5.1. □

4. Algebraic independence of Drinfeld logarithms

4.1. **Some linear algebraic groups.** For each $n \geq 0$, we denote by $G_{[n]}$ the $(4+2n)$-dimensional linear algebraic subgroup of GL_{2+n} over $\mathbb{F}_q(t)$:

$$G_{[n]} := \left\{ \begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{2+n} \right\}.$$

We let

$$\mathbf{X}_n := \begin{bmatrix} X_{11} & X_{12} & 0 & \cdots & 0 \\ X_{21} & X_{22} & 0 & \cdots & 0 \\ X_1 & Y_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_n & Y_n & 0 & \cdots & 1 \end{bmatrix}$$

be the coordinates of $G_{[n]}$. Throughout this section, we fix a positive integer m and consider a sequence of linear algebraic groups $\{\Gamma_{[n]}\}_{0 \leq n \leq m}$ over $\mathbb{F}_q(t)$ with the following properties:

- $\Gamma_{[0]} = \mathrm{GL}_2$;
- for each $1 \leq n \leq m$, $\Gamma_{[n]} \subseteq G_{[n]}$;
- $\Gamma_{[n]}$ is absolutely irreducible;
- we have a surjective morphism $\pi_n : \Gamma_{[n]} \twoheadrightarrow \Gamma_{[n-1]}$, which coincides with the projection map on the upper left $(n+1) \times (n+1)$ square of elements in $\Gamma_{[n]}$.

In §4.3 we will specify the particular groups $\Gamma_{[n]}$ that we have in mind, but for now we need only that they satisfy the above properties.

Definition. For each $n \geq 0$, $\Gamma_{[n]}$ is said to have *full dimension* if $\dim \Gamma_{[n]} = 4 + 2n$.

Lemma 4.1.1. *Suppose $\{\Gamma_{[n]}\}_{0 \leq n \leq m}$ is defined as above. For $1 \leq n \leq m$, if $\Gamma_{[n-1]}$ has full dimension, then*

$$\dim \Gamma_{[n]} = \dim \Gamma_{[n-1]} \quad \text{or} \quad \dim \Gamma_{[n]} = \dim \Gamma_{[n-1]} + 2.$$

Proof. We consider the short exact sequence of linear algebraic groups

$$1 \to V \to \Gamma_{[n]} \overset{\pi_n}{\twoheadrightarrow} \Gamma_{[n-1]} \to 1.$$

We note that

$$V \subseteq \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & 0 & \cdots & 1 \end{bmatrix} \right\} \subseteq \mathrm{GL}_{2+n}$$

and hence V has the natural structure of an additive group. Moreover, since $\Gamma_{[n-1]}$ has full dimension, for any $a \in \overline{\mathbb{F}_q(t)}^{\times}$ we can pick $\gamma \in \Gamma_{[n]}(\overline{\mathbb{F}_q(t)})$ so that $\pi_n(\gamma)$ is the block diagonal matrix

$$\pi_n(\gamma) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \oplus \mathrm{Id}_{n-1} \in \Gamma_{[n-1]}(\overline{\mathbb{F}_q(t)}).$$

Direct calculation shows that $\gamma^{-1} v \gamma \in V(\overline{\mathbb{F}_q(t)})$ for $v \in V(\overline{\mathbb{F}_q(t)})$ and hence V is a vector group.

We need only consider the case that $\dim V = 1$. We claim that in that case

$$V \subseteq \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \cdots & 1 \end{bmatrix} \right\} \subseteq \mathrm{GL}_{2+n} \text{ or } V \subseteq \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & 0 & \cdots & 1 \end{bmatrix} \right\} \subseteq \mathrm{GL}_{2+n}.$$

If the claim does not hold, then there exists $v \in V(\overline{\mathbb{F}_q(t)})$ with both the $(n+2,1)$ and $(n+2,2)$ entries non-zero. Take $a \in \overline{\mathbb{F}_q(t)}$, $a \neq 0$, 1, and pick any $\delta \in \Gamma_{[n]}(\overline{\mathbb{F}_q(t)})$ so that $\pi_n(\delta)$ is the block diagonal matrix

$$\pi_n(\delta) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \oplus \mathrm{Id}_{n-1} \in \Gamma_{[n-1]}(\overline{\mathbb{F}_q(t)}).$$

Then we see that $\delta^{-1} v \delta$ and v are $\overline{\mathbb{F}_q(t)}$-linearly independent vectors in $V(\overline{\mathbb{F}_q(t)})$, which contradicts $\dim V = 1$.

Now we pick $\eta \in \Gamma_{[n]}(\overline{\mathbb{F}_q(t)})$ so that $\pi_n(\eta)$ is the block diagonal matrix

$$\pi_n(\eta) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \mathrm{Id}_{n-1} \in \Gamma_{[n-1]}(\overline{\mathbb{F}_q(t)}).$$

Then the inclusion $\eta^{-1} V(\overline{\mathbb{F}_q(t)}) \eta \subseteq V(\overline{\mathbb{F}_q(t)})$ shows that

$$V \nsubseteq \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \cdots & 1 \end{bmatrix} \right\} \text{ and } V \nsubseteq \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & 0 & \cdots & 1 \end{bmatrix} \right\}.$$

Hence, the dimension of V is either 0 or 2. The proof is completed from the equality $\dim \Gamma_{[n]} = \dim \Gamma_{[n-1]} + \dim V$. □

Now suppose that $\Gamma_{[n-1]}$ has full dimension for some $1 \leq n \leq m$. We define the following one-dimensional subgroups of $\Gamma_{[n-1]} \subseteq \mathrm{GL}_{n+1}$:

$$T_1 := \left\{ \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right\}, \quad T_2 := \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right\},$$

$$U_1 := \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right\}, \quad U_2 := \left\{ \begin{bmatrix} 1 & * & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right\},$$

$$G_1 := \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ * & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right\}, \quad H_1 := \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & * & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right\},$$

$$\vdots$$

$$G_{n-1} := \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ * & 0 & 0 & \cdots & 1 \end{bmatrix} \right\}, \quad H_{n-1} := \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & * & 0 & \cdots & 1 \end{bmatrix} \right\}.$$

If $n = 1$, we define only T_i, U_i for $i = 1, 2$, without G_j, H_j.

Proposition 4.1.2. *Suppose $\{\Gamma_{[n]}\}_{0 \leq n \leq m}$ are defined as above, and suppose that $\Gamma_{[n-1]}$ has full dimension for some $1 \leq n \leq m$. If $\Gamma_{[n]}$ does not have full dimension, then π_n induces an isomorphism on $\mathbb{F}_q(t)$-rational points $\pi_n : \Gamma_{[n]}(\mathbb{F}_q(t)) \xrightarrow{\sim} \Gamma_{[n-1]}(\mathbb{F}_q(t))$.*

Proof. If we show that the induced tangent map

$$\mathrm{d}\pi_n : \operatorname{Lie} \Gamma_{[n]} \to \operatorname{Lie} \Gamma_{[n-1]}$$

is a surjective Lie algebra homomorphism, then we have that $\operatorname{Ker} \pi_n$ is defined over $\mathbb{F}_q(t)$ (see [14, Cor. 12.1.3]). Furthermore, by Lemma 4.1.1, $\operatorname{Ker} \pi_n$ is a zero-dimensional linear space over $\mathbb{F}_q(t)$ and hence [14, Prop. 12.3.4] implies the isomorphism $\Gamma_{[n]}(\mathbb{F}_q(t)) \cong \Gamma_{[n-1]}(\mathbb{F}_q(t))$ induced by π_n.

Let T_i, U_i, G_j, H_j be defined as above for $i = 1, 2$, $j = 1, \ldots, n-1$, and note that the Lie algebras of these $2n+2$ algebraic groups span $\operatorname{Lie} \Gamma_{[n-1]}$. To show the surjection of $\mathrm{d}\pi_n$, we need only construct one-dimensional algebraic subgroups T_i', U_i', G_j', H_j', of $\Gamma_{[n]}$ so that

$$T_i' \cong T_i, \quad U_i' \cong U_i, \quad G_j' \cong G_j, \quad H_j' \cong H_j, \quad \text{via } \pi_n,$$

for $i = 1, 2$, $j = 1, \ldots, n-1$. Then $\mathrm{d}\pi_n : \operatorname{Lie} \Gamma_{[n]} \twoheadrightarrow \operatorname{Lie} \Gamma_{[n-1]}$ is surjective since $\operatorname{Lie}(\cdot)$ is a left exact functor of algebraic groups.

Since $\operatorname{Ker} \pi_n$ is a zero-dimensional vector group, π_n is injective on points. Using this property, one checks directly that

- the Y_n-coordinates of $\pi_n^{-1}(T_1)$, $\pi_n^{-1}(U_1)$, $\pi_n^{-1}(G_1), \ldots, \pi_n^{-1}(G_{n-1})$, are zero;
- the X_n-coordinates of $\pi_n^{-1}(T_2)$, $\pi_n^{-1}(U_2)$, $\pi_n^{-1}(H_1), \ldots, \pi_n^{-1}(H_{n-1})$ are zero.

To construct T_1', T_2', we let $a, b \in \overline{\mathbb{F}_q(t)}^{\times} \setminus \overline{\mathbb{F}_q}^{\times}$ and pick $\gamma_1, \gamma_2 \in \Gamma_{[n]}(\overline{\mathbb{F}_q(t)})$ so that

$$\pi_n(\gamma_1) = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \text{ and } \pi_n(\gamma_2) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We let T_i' be the Zariski closure of the cyclic subgroup of $\Gamma_{[n]}$ generated by γ_i (inside $\Gamma_{[n]}$) for $i = 1, 2$. Then one checks directly that T_i' is a one-dimensional torus in $\Gamma_{[n]}$, and the restriction of π_n to T_i' induces an isomorphism $T_i' \cong T_i$ for $i = 1, 2$, (cf. [12, §6.2.4]). More precisely, the defining equations of T_1', T_2' can be written as follows:

$$T_1' : \left\{ \begin{array}{ll} (a-1)X_n - c(X_{11}-1) = 0, & X_{12} = 0 \\ X_{21} = 0, & X_{22} - 1 = 0 \\ X_1 = 0, & Y_1 = 0 \\ \vdots & \vdots \\ X_{n-1} = 0, & Y_n = 0 \end{array} \right\}, \tag{13}$$

$$T_2' : \left\{ \begin{array}{ll} X_{11} - 1 = 0, & X_{12} = 0 \\ X_{21} = 0, & (b-1)Y_n - d(X_{22}-1) = 0 \\ X_1 = 0, & Y_1 = 0 \\ \vdots & \vdots \\ X_n = 0, & Y_{n-1} = 0 \end{array} \right\}, \tag{14}$$

where c is the $(n+2,1)$-entry of γ_1 and d is the $(n+2,2)$-entry of γ_2.

For the constructions of U_1', U_2', we let $u_i \in U(\overline{\mathbb{F}_q(t)})$ be an $\mathbb{F}_q(t)$-rational basis for the one-dimensional vector group U_i and pick $u_i' \in \Gamma_{[n]}(\overline{\mathbb{F}_q(t)})$ so that $\pi_n(u_i') = u_i$ for $i = 1, 2$. We define U_i' to be the one-dimensional vector group in $\Gamma_{[n]}$ via the conjugations

$$\eta_1^{-1} u_1 \eta_1, \quad \eta_2^{-1} u_2 \eta_2, \quad \text{for } \eta_i \in T_i', \ i = 1, 2.$$

Then we see that $U_i' \cong U_i$ via π_n for $i = 1, 2$.

Finally we use the method above as well as conjugations to construct the desired G_j', H_j' so that $G_j' \cong G_j$ and $H_j' \cong H_j$ via π_n, for $j = 1, \dots, n-1$. The arguments are essentially the same as the constructions of T_i' and U_i', and we omit the details. □

4.2. **Defining equations for $\Gamma_{[n]}$.** We continue with the notation of the previous section, and we assume that $\Gamma_{[n-1]}$ has full dimension for some $1 \leq n \leq m$. Furthermore, we assume that $\Gamma_{[n]}$ does not have full dimension, and so by Lemma 4.1.1, $\dim \Gamma_{[n]} = \dim \Gamma_{[n-1]}$. Since we have shown that $\pi_n : \Gamma_{[n]}(\mathbb{F}_q(t)) \xrightarrow{\sim} \Gamma_{[n-1]}(\mathbb{F}_q(t))$, for any a, $b \in \mathbb{F}_q(t)^\times \setminus \mathbb{F}_q^\times$ we can pick γ_1, $\gamma_2 \in \Gamma_{[n]}(\mathbb{F}_q(t))$ so that

$$\pi_n(\gamma_1) = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \text{ and } \pi_n(\gamma_2) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If we let T_i' be the Zariski closure of the cyclic group generated by γ_i (inside $\Gamma_{[n]}$), then the defining equations of T_i' are given as in (13) and (14) for $i = 1, 2$. We note that c and d in (13) and (14) are in $\mathbb{F}_q(t)$.

Let V_1 be the n-dimensional vector group over $\mathbb{F}_q(t)$ in $\Gamma_{[n-1]}$ spanned by $U_1, G_1, \dots, G_{n-1}$ given as above. Let $U_1', G_1', \dots, G_{n-1}'$ be the one-dimensional vector groups in $\Gamma_{[n]}$ given as in the proof of Proposition 4.1.2. Let V_1' be the n-dimensional vector group over $\mathbb{F}_q(t)$ in $\Gamma_{[n]}$ spanned by $U_1', G_1', \dots, G_{n-1}'$.

Similar to the constructions above, we let V_2 be the n-dimensional vector group over $\mathbb{F}_q(t)$ in $\Gamma_{[n-1]}$ spanned by $U_2, H_1, \dots, H_{n-1}$. Let $U_2', H_1', \dots, H_{n-1}'$ be the one-dimensional vector groups in $\Gamma_{[n]}$ given as in the proof of Proposition 4.1.2. We define V_2' to be the n-dimensional vector group over $\mathbb{F}_q(t)$ in $\Gamma_{[n]}$ spanned by $U_2', H_1', \dots, H_{n-1}'$ and note that V_i' is isomorphic to V_i via π_n for $i = 1, 2$.

Note that the defining equations of V_1', V_2' are given as follows:

$$V_1' : \left\{ \begin{array}{l} X_{11} - 1 = 0,\ r_{21}X_{21} + r_1X_1 + \cdots + r_nX_n = 0 \\ X_{12} = 0,\ X_{22} - 1 = 0,\ Y_1 = 0, \dots, Y_n = 0 \end{array} \right\},$$
$$V_2' : \left\{ \begin{array}{l} X_{11} - 1 = 0,\ X_{21} = 0,\ X_1 = 0, \dots, X_n = 0 \\ X_{22} - 1 = 0,\ s_{12}X_{12} + s_1Y_1 + \cdots + s_nY_n = 0 \end{array} \right\},$$

for some $r_{21}, r_1, \dots, r_n, s_{12}, s_1, \dots, s_n \in \mathbb{F}_q(t)$. Note that, since V_i' is isomorphic to V_i via π_n for $i = 1, 2$, we have that $r_n \neq 0$, $s_n \neq 0$.

We define P_i' to be the Zariski closure of the subgroup generated by T_i' and V_i' inside $\Gamma_{[n]}$ for $i = 1, 2$. Then we see that for each $i = 1, 2$, P_i' is the $(n+1)$-dimensional affine linear space containing T_i' and V_i' (cf. [12, §6.2.4]), and hence their defining equations can be described as follows:

$$\begin{aligned} &P_1' : \left\{ \begin{array}{l} \phi_1 := (a-1)(r_{21}X_{21} + r_1X_1 + \cdots + r_nX_n) - cr_n(X_{11} - 1) = 0, \\ X_{12} = 0,\ X_{22} - 1 = 0,\ Y_1 = 0, \dots, Y_n = 0 \end{array} \right\}, \\ &P_2' : \left\{ \begin{array}{l} X_{11} - 1 = 0,\ X_{21} = 0, X_1 = 0, \dots, X_n = 0, \\ \phi_2 := (b-1)(s_{12}X_{12} + s_1Y_1 + \cdots + s_nY_n) - ds_n(X_{22} - 1) = 0 \end{array} \right\}. \end{aligned} \tag{15}$$

Note that $\dim P_i' = n + 1$ for $i = 1, 2$. Consider now the morphism defined by the product of matrices,

$$P_1' \times P_2' \to \Gamma_{[n]}.$$

Its image is denoted by $P_1' \cdot P_2'$. We claim that the Zariski closure $\overline{P_1' \cdot P_2'}$ of $P_1' \cdot P_2'$ inside $\Gamma_{[n]}$ is all of $\Gamma_{[n]}$.

To prove this claim, we define P_i to be the Zariski closure of the subgroup of $\Gamma_{[n-1]}$ generated by T_i and V_i for $i = 1, 2$. That is, since $\Gamma_{[n-1]}$ has full dimension,

$$P_1 := \left\{ \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ * & 0 & 0 & \cdots & 1 \end{bmatrix} \right\} \subseteq G_{[n-1]}, \quad P_2 := \left\{ \begin{bmatrix} 1 & * & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & * & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & * & 0 & \cdots & 1 \end{bmatrix} \right\} \subseteq G_{[n-1]}.$$

Direct calculation shows that

$$G_{[n-1]} = \overline{P_1 \cdot P_2} \cup V(X_{11}),$$

where $V(X_{11})$ is the closed subvariety of $G_{[n-1]}$ given by $X_{11} = 0$. Hence, $\Gamma_{[n-1]} = \overline{P_1 \cdot P_2}$ since $\Gamma_{[n-1]} \subseteq G_{[n-1]}$ is assumed to be irreducible of maximal dimension.

Furthermore, since we have a surjective map $P_i' \twoheadrightarrow P_i$ induced by π_n for $i = 1, 2$, the restriction of π_n to $\overline{P_1' \cdot P_2'}$ is dominant and hence $\dim \overline{P_1' \cdot P_2'} \geq 2 + 2n$. From the assumptions that $\dim \Gamma_{[n]} = \dim \Gamma_{[n-1]} = 2 + 2n$ and that $\Gamma_{[n]}$ is irreducible, we see that $\overline{P_1' \cdot P_2'} = \Gamma_{[n]}$.

On the other hand, one can similarly show that $\overline{P_2' \cdot P_1'} = \Gamma_{[n]}$. We omit the details.

Now, we claim that ϕ_1, ϕ_2, as defined in (15), give rise to defining equations for $\Gamma_{[n]}$. For polynomials $g_1, \dots, g_m \in \mathbb{F}_q(t)[\mathbf{X}_n]$, we let $V(g_1, \dots, g_m)$ be the closed subvariety of $G_{[n]}$

given by $g_1 = \cdots = g_m = 0$. Given any

$$(16) \qquad \mathbf{p}_1 = \begin{bmatrix} x_{11} & 0 & 0 & \cdots & 0 \\ x_{21} & 1 & 0 & \cdots & 0 \\ x_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ x_n & 0 & 0 & \cdots & 1 \end{bmatrix} \in P_1' \text{ and } \mathbf{p}_2 = \begin{bmatrix} 1 & x_{12} & 0 & \cdots & 0 \\ 0 & x_{22} & 0 & \cdots & 0 \\ 0 & y_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & y_2 & 0 & \cdots & 1 \end{bmatrix} \in P_2',$$

from the matrix product,

$$\mathbf{p}_1\mathbf{p}_2 = \begin{bmatrix} x_{11} & * & 0 & \cdots & 0 \\ x_{21} & * & 0 & \cdots & 0 \\ x_1 & * & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ x_n & * & 0 & \cdots & 1 \end{bmatrix} \in P_1' \cdot P_2',$$

we see that $V(\phi_1) \supseteq P_1' \cdot P_2'$, and hence $V(\phi_1) \supseteq \Gamma_{[n]}$. A similar calculation using the matrix product $\mathbf{p}_2\mathbf{p}_1 \in P_2' \cdot P_1'$ shows that $V(\phi_2) \supseteq P_2' \cdot P_1'$. Furthermore, because ϕ_1, ϕ_2 are degree one polynomials, $V(\phi_1, \phi_2)$ is irreducible, and therefore $V(\phi_1, \phi_2) = \Gamma_{[n]}$, as claimed.

Let $\phi_1 := \ell_{11}X_{11} + \ell_{21}X_{21} + \ell_1 X_1 + \cdots + \ell_n X_n - \ell_{11}$, where

$$\ell_{11} = -cr_n, \quad \ell_{21} = (a-1)r_{21}, \ell_i = (a-1)r_i, \quad i = 1, \ldots, n.$$

Note that all coefficients of ϕ_1 are in $\mathbb{F}_q(t)$ and $\ell_n \neq 0$. Finally we claim that, without loss of generality, ϕ_2 can be written as

$$\phi_2 = \ell_{11}X_{12} + \ell_{21}X_{22} + \ell_1 Y_1 + \cdots + \ell_n Y_n - \ell_{21}.$$

To verify the claim, we note that, modulo $(b-1)$, without loss of generality we may write ϕ_2 as $\phi_2 = s_{12}X_{12} + s_{22}X_{22} + \sum_{i=1}^n s_i Y_i - s_{22}$, where $s_{22} := \frac{-ds_n}{b-1}$. Choose an element of the form

$$\mathbf{p}_2 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & y_n & 0 & \cdots & 1 \end{bmatrix} \in P_2'(\overline{\mathbb{F}_q(t)}), \quad \text{i.e., } \phi_2(\mathbf{p}_2) = 0.$$

We choose an arbitrary $\mathbf{p}_1 \in P_1'(\overline{\mathbb{F}_q(t)})$ as in (16), and because $\phi_1(\mathbf{p}_1) = 0$ and $\mathbf{p}_1\mathbf{p}_2 \in V(\phi_1, \phi_2)$, a direct calculation shows that

$$s_{12}x_{11} + s_{22}x_{21} + \sum_{i=1}^n s_i x_i - s_{12} = 0.$$

Namely, P_1' is contained in the $(n+1)$-dimensional affine linear space given by

$$V\left(s_{12}X_{11} + s_{22}X_{21} + \sum_{i=1}^n s_i X_i - s_{12},\ X_{12},\ X_{22} - 1,\ Y_1, \ldots, Y_n \right).$$

Since P_1' is also an affine linear space of dimension $n+1$, we see that the two vectors

$$(\ell_{11}, \ell_{21}, \ell_1, \ldots, \ell_n), \quad (s_{12}, s_{22}, s_1, \cdots, s_n)$$

are parallel, which completes the claim. We summarize the investigations of this section in the following lemma.

Lemma 4.2.1. *Let $\{\Gamma_{[n]}\}_{0\leq n\leq m}$ be a sequence of groups defined as in §4.1. Suppose that $\Gamma_{[n-1]}$ has full dimension for some $1 \leq n \leq m$ but that $\dim \Gamma_{[n]} = \dim \Gamma_{[n-1]}$. Then there exist $\ell_{11}, \ell_{21}, \ell_1, \cdots, \ell_n \in \mathbb{F}_q(t)$ with $\ell_n \neq 0$ so that*

$$\phi_1 := \ell_{11}X_{11} + \ell_{21}X_{21} + \ell_1 X_1 + \cdots + \ell_n X_n - \ell_{11},$$
$$\phi_2 := \ell_{11}X_{12} + \ell_{21}X_{22} + \ell_1 Y_1 + \cdots + \ell_n Y_n - \ell_{21}$$

are defining polynomials for $\Gamma_{[n]}$.

4.3. **Application to Drinfeld logarithms.** We fix a rank 2 Drinfeld $\mathbb{F}_q[t]$-module ρ over $\overline{k}$ with $\rho_t := \theta + \kappa\tau + \tau^2$, $\kappa \in \overline{k}$, and we fix an $\mathbb{F}_q[\theta]$-basis $\{\omega_1, \omega_2\}$ of the period lattice $\Lambda_\rho := \operatorname{Ker} \exp_\rho$. We let $\Phi := \Phi_\rho$, $\Psi := \Psi_\rho$, ξ, and Ω be defined as in §2.4.

Given $\lambda \in \mathbb{C}_\infty$ with $\exp_\rho(\lambda) =: \alpha \in \overline{k}$, let f_λ be the Anderson generating function associated to λ as in (7). We define

$$\mathbf{g} := \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} := \begin{bmatrix} -\kappa f_\lambda^{(1)} - f_\lambda^{(2)} \\ -f_\lambda^{(1)} \end{bmatrix} \in \operatorname{Mat}_{2\times 1}(\mathbb{T}).$$

By (8) and (9) we see that

$$\Phi^{\mathrm{tr}}\mathbf{g}^{(-1)} = \mathbf{g} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad g_1(\theta) = \lambda - \alpha, \quad g_2(\theta) = -F_\tau(\lambda). \tag{17}$$

Given $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ with $\exp_\rho(\lambda_i) =: \alpha_i \in \overline{k}$ for $i = 1, \ldots, m$, for each $1 \leq n \leq m$ let

$$\mathbf{g}_n := \begin{bmatrix} g_{n1} \\ g_{n2} \end{bmatrix} := \begin{bmatrix} -\kappa f_{\lambda_n}^{(1)} - f_{\lambda_n}^{(2)} \\ -f_{\lambda_n}^{(1)} \end{bmatrix} \quad \text{and} \quad \mathbf{h}_n := \begin{bmatrix} \alpha_n \\ 0 \end{bmatrix}.$$

We further define

$$\Phi_n := \begin{bmatrix} \Phi & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{h}_1^{\mathrm{tr}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{h}_n^{\mathrm{tr}} & 0 & \cdots & 1 \end{bmatrix} \in \operatorname{Mat}_{2+n}(\overline{k}[t]), \quad \Psi_n := \begin{bmatrix} \Psi & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{g}_1^{\mathrm{tr}}\Psi & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_n^{\mathrm{tr}}\Psi & 0 & \cdots & 1 \end{bmatrix} \in \operatorname{GL}_{2+n}(\mathbb{T}).$$

Using (17) we have $\Psi_n^{(-1)} = \Phi_n \Psi_n$ and thus

$$\overline{k}(\Psi_n(\theta)) = \overline{k}(\omega_1, \omega_2, \lambda_1, \ldots, \lambda_n, F_\tau(\omega_1), F_\tau(\omega_2), F_\tau(\lambda_1), \ldots, F_\tau(\lambda_n)). \tag{18}$$

Proposition 4.3.1. *For each $1 \leq n \leq m$, let Φ_n be defined as above. Then Φ_n defines a t-motive M_n.*

Proof. By Lemma 2.4.1, Φ itself defines an Anderson t-motive $\mathcal{M}_\rho$. Using this the proof is then essentially the same as the proof of [12, Prop. 6.1.3]. We omit the details. □

Suppose that ρ does not have complex multiplication. Let $M_0 := M_\rho$ be the t-motive associated to ρ (see §2.4), and let $\Gamma_{[0]} := \Gamma_{M_0} = \operatorname{GL}_2$ be its Galois group (see Theorem 3.4.1). For each $1 \leq n \leq m$, let $\Gamma_{[n]}$ be the Galois group of M_n. By (5), we see that $\Gamma_{[n]} \subseteq G_{[n]}$. Moreover, since M_{n-1} is a sub-t-motive of M_n, we have a surjective map

$$\pi_n : \Gamma_{[n]} \twoheadrightarrow \Gamma_{[n-1]}$$

(cf. proof of Proposition 3.1.2). More precisely, for any $\mathbb{F}_q(t)$-algebra R the restriction of the action of any $\gamma \in \Gamma_{[n]}(R)$ to $R \otimes_{\mathbb{F}_q(t)} M_{n-1}^B$ is the same as the action of the upper left $(n+1) \times (n+1)$ square of γ. We see that the map π_n coincides with the projection map on the upper left $(n+1)\times(n+1)$ square of elements in $\Gamma_{[n]}$. Thus, the sequence $\{\Gamma_{[n]}\}_{0\le n\le m}$ satisfies the defining conditions of §4.1, and the results of §4.1 and §4.2 apply.

Finally, by Theorem 2.2.1 we note that $\dim \Gamma_{[n]} = \operatorname{tr.deg}_{\overline{k}} \overline{k}(\Psi_n(\theta))$ for each $0 \le n \le m$. Combining Theorem 3.4.1, Lemma 4.1.1, and (18), we now prove a lemma that is the heart of Theorem 4.3.3 which follows.

Lemma 4.3.2. *Suppose that p is odd. Let ρ be a Drinfeld $\mathbb{F}_q[t]$-module without complex multiplication with $\rho_t = \theta + \kappa\tau + \tau^2$, $\kappa \in \overline{k}$. Let $\Lambda_\rho := \mathbb{F}_q[\theta]\omega_1 + \mathbb{F}_q[\theta]\omega_2$ be the period lattice of ρ. Suppose that $\lambda_1,\dots,\lambda_m \in \mathbb{C}_\infty$ satisfy $\exp_\rho(\lambda_i) =: \alpha_i \in \overline{k}$ for $i=1,\dots,m$ and that $\omega_1,\omega_2,\lambda_1,\dots,\lambda_m$ are linearly independent over k. Finally, let $\{\Gamma_{[n]}\}_{0\le n\le m}$ be defined as above. If $\Gamma_{[n-1]}$ has full dimension for some $1\le n\le m$, then so does $\Gamma_{[n]}$. In particular, the $4+2m$ elements*

$$\omega_1,\omega_2,\lambda_1,\dots,\lambda_m, F_\tau(\omega_1), F_\tau(\omega_2), F_\tau(\lambda_1),\dots,F_\tau(\lambda_m)$$

are algebraically independent over $\overline{k}$.

Proof. Since $\Gamma_{[0]} = \mathrm{GL}_2$, we see that $\Gamma_{[n-1]}$ has full dimension when $n=1$. Thus we assume for arbitrary n that $\Gamma_{[n-1]}$ has full dimension but that $\Gamma_{[n]}$ does not. Then by Lemma 4.1.1 we have $\dim\Gamma_{[n]} = \dim\Gamma_{[n-1]}$. Moreover, by Lemma 4.2.1 there exist ℓ_{11}, ℓ_{21}, $\ell_1,\dots,\ell_n \in \mathbb{F}_q(t)$ with $\ell_n \neq 0$ so that

$$\phi_1 := \ell_{11}X_{11} + \ell_{21}X_{21} + \ell_1X_1 + \cdots + \ell_nX_n - \ell_{11},$$
$$\phi_2 := \ell_{11}X_{12} + \ell_{21}X_{22} + \ell_1Y_1 + \cdots + \ell_nY_n - \ell_{21},$$

are defining polynomials for $\Gamma_{[n]}$.

By Theorem 2.3.1, Z_{Ψ_n} is a torsor for $\Gamma_{[n]} \times_{\mathbb{F}_q(t)} \overline{k}(t)$ over $\overline{k}(t)$. Since $\Gamma_{[n]}$ is an affine linear space over $\mathbb{F}_q(t)$, Z_{Ψ_n} is also an affine linear space over $\overline{k}(t)$. Hence $Z_{\Psi_n}(\overline{k}(t))$ is non-empty. Choosing $\delta \in Z_{\Psi_n}(\overline{k}(t))$, we see that $Z_{\Psi_n}(\overline{k}(t)) = \delta\cdot\Gamma_{\Psi_n}(\overline{k}(t))$, which implies that

$$AX_{11} + BX_{21} + \ell_1X_1 + \cdots + \ell_nX_n - \ell_{11}, \quad AX_{12} + BX_{22} + \ell_1Y_1 + \cdots + \ell_nY_n - \ell_{21},$$

are defining polynomials for Z_{Ψ_n}, where

$$[A,B,\ell_1,\dots,\ell_n] := [\ell_{11},\ell_{21},\ell_1,\dots,\ell_n]\delta^{-1}.$$

We claim that A, $A^{(-1)}$, B, $B^{(-1)} \in \overline{k}(t)$ are regular at $t=\theta$. Let $[F_i,G_i] := \mathbf{g}_i^{\mathrm{tr}}\Psi$ for $i=1,\dots,n$. Then by the definition of Z_{Ψ_n} in §2.3 we have two equations

$$A\Psi_{11} + B\Psi_{21} + \ell_1F_1 + \cdots + \ell_nF_n - \ell_{11} = 0, \tag{19}$$
$$A\Psi_{12} + B\Psi_{22} + \ell_1G_1 + \cdots + \ell_nG_n - \ell_{21} = 0. \tag{20}$$

Using the σ-action on (19) and then subtracting it from itself, we have:

$$\left(A - B^{(-1)}(t-\theta) - \sum_{i=1}^n \alpha_i\ell_i\right)\Psi_{11} + \left(B - A^{(-1)} + \kappa^{(-1)}B^{(-1)}\right)\Psi_{21} = 0.$$

Since $\Gamma_{[0]} = \Gamma_\Psi = \mathrm{GL}_2$, Theorem 2.3.1 implies that the four functions $\{\Psi_{ij} : i=1,2,\ j=1,2\}$ are algebraically independent over $\overline{k}(t)$. Hence

$$A - B^{(-1)}(t-\theta) - \sum_{i=1}^n \alpha_i\ell_i = 0, \quad B - A^{(-1)} + \kappa^{(-1)}B^{(-1)} = 0, \tag{21}$$

and thus,

$$B+\kappa^{(-1)}B^{(-1)}-(t-\theta^{(-1)})B^{(-2)}=\sum_{i=1}^{n}\alpha_i^{(-1)}\ell_i.$$

Note that the right hand side of this equation is regular at $t=\theta^{q^i}$ for all $i\in\mathbb{Z}$. Indeed if B has a pole at $t=\theta$, then either $B^{(-1)}$ or $B^{(-2)}$ has pole at $t=\theta$. That is, either B has pole at $t=\theta^q$ or B has pole at $t=\theta^{q^2}$. Continuing this argument, we see that B has infinitely many poles among $\{t=\theta^{q^i}\}_{i=1}^{\infty}$, which contradicts that $B\in\overline{k}(t)$. Using a similar argument, we see that $B^{(-1)}$ is also regular at $t=\theta$. By (21), we thus see that A and $A^{(-1)}$ are regular at $t=\theta$ and that

$$A(\theta)=\sum_{i=1}^{n}\alpha_i\ell_i(\theta). \tag{22}$$

Recall that $\Psi(\theta)$ is given explicitly in (11). By specializing (19) and (20) at $t=\theta$ and using (22), we have

$$\Bigl(\sum_{i=1}^{n}\ell_i(\theta)\lambda_i\Bigr)\xi F_\tau(\omega_2)+\Bigl(B(\theta)-\sum_{i=1}^{n}\ell_i(\theta)F_\tau(\lambda_i)\Bigr)\xi\omega_2-\ell_{11}(\theta)\widetilde{\pi}=0, \tag{23}$$

$$-\Bigl(\sum_{i=1}^{n}\ell_i(\theta)\lambda_i\Bigr)\xi F_\tau(\omega_1)-\Bigl(B(\theta)-\sum_{i=1}^{n}\ell_i(\theta)F_\tau(\lambda_i)\Bigr)\xi\omega_1-\ell_{21}(\theta)\widetilde{\pi}=0. \tag{24}$$

Using the analogue of the Legendre relation (10), then (23) $\times\,\omega_1$ + (24) $\times\,\omega_2=0$ implies

$$\ell_1(\theta)\lambda_1+\cdots+\ell_n(\theta)\lambda_n-\ell_{11}(\theta)\omega_1-\ell_{21}(\theta)\omega_2=0.$$

Since $\ell_n(\theta)\neq 0$, we obtain a non-trivial k-linear dependence among ω_1, ω_2, $\lambda_1,\dots,\lambda_n$, which is a contradiction. □

Theorem 4.3.3. *Suppose that p is odd. Let ρ be any rank 2 Drinfeld $\mathbb{F}_q[t]$-module defined over $\overline{k}$ without complex multiplication. Let $\lambda_1,\dots,\lambda_m\in\mathbb{C}_\infty$ satisfy $\exp_\rho(\lambda_i)\in\overline{k}$ for $i=1,\dots,m$. If $\lambda_1,\dots,\lambda_m$ are linearly independent over k, then the $2m$ elements*

$$\lambda_1,\dots,\lambda_m,F_\tau(\lambda_1),\dots,F_\tau(\lambda_m)$$

are algebraically independent over $\overline{k}$.

Proof. By Remark 3.4.2 we may assume without loss of generality that the coefficient of τ^2 in ρ_t is 1. Let $\Lambda_\rho:=\mathbb{F}_q[\theta]\omega_1+\mathbb{F}_q[\theta]\omega_2$ be the period lattice of ρ. Let $\mathbf{\Lambda}_\rho$ be the k-vector space spanned by ω_1 and ω_2, and let $\{v_1,v_2\}$ be any k-basis of $\mathbf{\Lambda}_\rho$. Given any $z_1,\dots,z_n\in\mathbb{C}_\infty$ such that $\exp_\rho(z_i)\in\overline{k}$ for $i=1,\dots,n$, we observe that

$$k\text{-Span}\{\omega_1,\omega_2,z_1,\dots,z_n\}=k\text{-Span}\{v_1,v_2,z_1,\dots,z_n\}; \tag{25}$$

$$\overline{k}\bigl(\cup_{j=1}^{n}\cup_{i=1}^{2}\{\omega_i,F_\tau(\omega_i),z_j,F_\tau(z_j)\}\bigr)=\overline{k}\bigl(\cup_{j=1}^{n}\cup_{i=1}^{2}\{v_i,F_\tau(v_i),z_j,F_\tau(z_j)\}\bigr). \tag{26}$$

Note that (26) follows from the fact that the quasi-periodic function F_τ is $\mathbb{F}_q[\theta]$-linear on Λ_ρ and satisfies difference equations as in (1).

We define $N:=k\text{-Span}\{\omega_1,\omega_2,\lambda_1,\dots,\lambda_m\}$. By Lemma 4.3.2, the theorem is immediately true if $\dim_k N=m+2$. If $\dim_k N=m+1$, then without loss of generality we can assume that $\omega_2=b_0\omega_1+\sum_{j=1}^{m}b_j\lambda_j$, $b_j\in k$, with $b_1\neq 0$. In that case, $\{\omega_1,\sum b_j\lambda_j\}$ is a k-basis of $\mathbf{\Lambda}_\rho$ and $\{\omega_1,\sum b_j\lambda_j,\lambda_2,\dots\lambda_m\}$ is a k-basis of N. The result then follows from Lemma 4.3.2 combined with (25) and (26). If $\dim_k N=m$, then in a similar manner, we can find b_j,

$c_j \in k$ so that $\{\sum_j b_j\lambda_j, \sum_j c_j\lambda_j\}$ is a k-basis of $\mathbf{\Lambda}_\rho$ and $\{\sum_j b_j\lambda_j, \sum_j c_j\lambda_j, \lambda_3, \ldots, \lambda_m\}$ is a k-basis of N. Again the result follows from Lemma 4.3.2 combined with (25) and (26). □

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