THETA OPERATORS, GOSS POLYNOMIALS, AND v-ADIC MODULAR FORMS

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In honor of David Goss

Abstract. We investigate hyperderivatives of Drinfeld modular forms and determine formulas for these derivatives in terms of Goss polynomials for the kernel of the Carlitz exponential. As a consequence we prove that v-adic modular forms in the sense of Serre, as defined by Goss and Vincent, are preserved under hyperdifferentiation. Moreover, upon multiplication by a Carlitz factorial, hyperdifferentiation preserves v-integrality.

1. Introduction

In [21], Serre introduced p-adic modular forms for a fixed prime p, as p-adic limits of Fourier expansions of holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with rational coefficients. He established fundamental results about families of p-adic modular forms by developing the theories of differential operators and Hecke operators acting on p-adic spaces of modular forms, and in particular he showed that the weight 2 Eisenstein series E_2 is also p-adic. If we let $\vartheta := \frac{1}{2\pi i}\frac{d}{dz}$ be Ramanujan's theta operator acting on holomorphic complex forms, then letting $\mathbf{q}(z) = e^{2\pi i z}$, we have

$$\vartheta = \mathbf{q}\frac{d}{d\mathbf{q}}, \quad \vartheta(\mathbf{q}^n) = n\mathbf{q}^n. \tag{1.1}$$

Although ϑ does not preserve spaces of complex modular forms, Serre proved the induced operation $\vartheta : \mathbb{Q}\otimes\mathbb{Z}_p[[\mathbf{q}]] \to \mathbb{Q}\otimes\mathbb{Z}_p[[\mathbf{q}]]$ does take p-adic modular forms to p-adic modular forms and preserves p-integrality.

In the present paper we investigate differential operators on spaces of v-adic modular forms, where v is a finite place corresponding to a prime ideal of the polynomial ring $A = \mathbb{F}_q[\theta]$, for $\mathbb{F}_q$ a field with q elements and q itself a power of a prime number p. Drinfeld modular forms were first studied by Goss [10], [11], [12], as rigid analytic functions,

$$f : \Omega \to C_\infty,$$

on the Drinfeld upper half space Ω that transform with respect to the group $\Gamma = \mathrm{GL}_2(A)$ (see §4 for precise definitions). Here if we take $K = \mathbb{F}_q(\theta)$, then Ω is defined to be $C_\infty \setminus K_\infty$, where $K_\infty = \mathbb{F}_q((1/\theta))$ is the completion of K at its infinite place and C_∞ is the completion of an algebraic closure of K_∞. Goss showed that Drinfeld modular forms have expansions in terms of the uniformizing parameter $u(z) := 1/e_C(\widetilde{\pi}z)$ at the infinite

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Date: December 9, 2016.

2010 *Mathematics Subject Classification.* Primary 11F52; Secondary 11F33, 11G09.

Key words and phrases. Drinfeld modular forms, Goss polynomials, v-adic modular forms, hyperderivatives, false Eisenstein series.

This project was partially supported by NSF Grant DMS-1501362.

cusp of Ω, where $e_C(z)$ is the exponential function of the Carlitz module and $\widetilde{\pi}$ is the Carlitz period. Each such form f is uniquely determined by its u-expansion,

$$f = \sum_{n=0}^{\infty} c_n u^n \in C_\infty[[u]].$$

If $k \equiv 0 \pmod{q-1}$, then the weight k Eisenstein series of Goss [11], has a u-expansion due to Gekeler [7, (6.3)] of the form

$$E_k = -\frac{\zeta_C(k)}{\widetilde{\pi}^k} - \sum_{a \in A,\, a \text{ monic}} G_k(u(az)), \quad \frac{\zeta_C(k)}{\widetilde{\pi}^k} \in K,$$

where $\zeta_C(k)$ is a Carlitz zeta value, $G_k(u)$ is a Goss polynomial of degree k for the lattice $\Lambda_C = A\widetilde{\pi}$ (see §3–4 and (4.2)), and $u(az)$ can be shown to be represented as a power series in u (see §4). Gekeler and Goss also show that spaces of forms for Γ are generated by forms with u-expansions with coefficients in A. Using this as a starting point, Goss [14] and Vincent [24] defined v-adic modular forms in the sense of Serre by taking v-adic limits of u-expansions and thus defining v-adic forms as power series in $K \otimes_A A_v[[u]]$ (see §5). Goss [14] constructed a family of v-adic forms based on forms with A-expansions due to Petrov [19] (see Theorem 6.3), and Vincent [24] showed that forms for the group $\Gamma_0(v) \subseteq \mathrm{GL}_2(A)$ with v-integral u-expansions are also v-adic modular forms.

It is natural to ask how Drinfeld modular forms and v-adic forms behave under differentiation, and since we are in positive characteristic it is favorable to use hyperdifferential operators ∂_z^r, rather than straight iteration $\frac{d^r}{dz^r} = \frac{d}{dz} \circ \cdots \circ \frac{d}{dz}$ (see §2 for definitions). Gekeler [7, §8] showed that if we define $\Theta := -\frac{1}{\widetilde{\pi}}\frac{d}{dz} = -\frac{1}{\widetilde{\pi}}\,\partial_z^1$, then we have the action on u-expansions determined by the equality

$$\Theta = u^2 \frac{d}{du} = u^2\, \partial_u^1. \tag{1.2}$$

Now as in the classical case, derivatives of Drinfeld modular forms are not necessarily modular, but Bosser and Pellarin [1], [2], showed that hyperdifferential operators ∂_z^r preserve spaces of quasi-modular forms, i.e., spaces generated by modular forms and the false Eisenstein series E of Gekeler (see Example 4.7), which itself plays the role of E_2.

For $r \geqslant 0$, following Bosser and Pellarin we define the operator Θ^r by

$$\Theta^r := \frac{1}{(-\widetilde{\pi})^r}\, \partial_z^r.$$

Uchino and Satoh [22, Lem. 3.6] proved that Θ^r takes functions with u-expansions to functions with u-expansions, and Bosser and Pellarin [1, Lem. 3.5] determined formulas for the expansion of $\Theta^r(u^n)$. If we consider the r-th iterate of the classical ϑ-operator, $\vartheta^{\circ r} = \vartheta \circ \cdots \circ \vartheta$, then clearly by (1.1),

$$\vartheta^{\circ r}(q^n) = n^r q^n.$$

If we iterate Θ, taking $\Theta^{\circ r} = \Theta \circ \cdots \circ \Theta$, then by (1.2) we find

$$\Theta^{\circ r}(u^n) = r! \binom{n+r-1}{r} u^{n+r},$$

which vanishes identically when $r \geqslant p$. On the other hand, the factor of $r!$ is not the only discrepancy in comparing Θ^r and $\Theta^{\circ r}$, and in fact we prove two formulas in Corollary 4.10

revealing that Θ^r is intertwined with Goss polynomials for Λ_C:

$$\Theta^r(u^n) = u^n\,\partial_u^{n-1}\bigl(u^{n-2}G_{r+1}(u)\bigr), \quad \forall\, n \geqslant 1, \tag{1.3}$$

$$\Theta^r(u^n) = \sum_{j=0}^{r} \binom{n+j-1}{j} \beta_{r,j} u^{n+j}, \quad \forall\, n \geqslant 0, \tag{1.4}$$

where $\beta_{r,j}$ are the coefficients of $G_{r+1}(u)$. These formulas arise from general results (Theorem 3.4) on hyperderivatives of Goss polynomials for arbitrary $\mathbb{F}_q$-lattices in C_∞, which is the primary workhorse of this paper, and they induce formulas for hyperderivatives of u-expansions of Drinfeld modular forms (Corollary 4.12). It is important to note that (1.4) is close to a formula of Bosser and Pellarin [1, Eq. (28)], although the connections with coefficients of Goss polynomials appears to be new and the approaches are somewhat different.

Goss [14] defines the weight space of v-adic modular forms to be $\mathbb{S} = \mathbb{Z}/(q^d-1)\mathbb{Z} \times \mathbb{Z}_p$, where d is the degree of v, and if we take $\mathcal{M}_s^m \subseteq K \otimes_A A_v[[u]]$ to be the space of v-adic forms of weight $s \in \mathbb{S}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ (see §5), then we prove (Theorem 6.1) that Θ^r preserves spaces of v-adic modular forms,

$$\Theta^r : \mathcal{M}_s^m \to \mathcal{M}_{s+2r}^{m+r}, \quad r \geqslant 0.$$

Of particular importance here is proving that the false Eisenstein series E is a v-adic form (Theorem 6.5). Vincent [23, Thm. 1.2] showed that $\Theta(f)$ is congruent to a modular form modulo v, so we show that this congruence lifts to v-adic modular forms. For $r \geqslant q$, unlike in the classical case, Θ^r does not preserve v-integrality due to denominators coming from $G_{r+1}(u)$, but we show in §7 that this failure can be controlled, namely showing (Theorem 7.4) that

$$\Pi_r\Theta^r : \mathcal{M}_s^m(A_v) \to \mathcal{M}_{s+2r}^{m+r}(A_v), \quad r \geqslant 0,$$

where $\Pi_r \in A$ is the Carlitz factorial (see §2) and $\mathcal{M}_s^m(A_v) = \mathcal{M}_s^m \cap A_v[[u]]$.

Acknowledgments. The authors thank David Goss for a number of helpful suggestions on an earlier version of this article, and they thank Bruno Anglès for translating the abstract into French. The authors also thank the referee for useful suggestions.

2. Functions and hyperderivatives

Let $\mathbb{F}_q$ be the finite field with q elements, q a fixed power of a prime p. Let $A := \mathbb{F}_q[\theta]$ be a polynomial ring in one variable, and let $K := \mathbb{F}_q(\theta)$ be its fraction field. We let A_+ denote the monic elements of A, A_{d+} the monic elements of degree d, and $A_{(<d)}$ the elements of A of degree $< d$.

For each place v of K, we define an absolute value $|\cdot|_v$ and valuation ord_v, normalized in the following way. If v is a finite place, we fix $\wp \in A_+$ to be the monic generator of the prime ideal $\mathfrak{p}_v$ corresponding to v and we set $|\wp|_v = 1/q^{\deg \wp}$ and $\mathrm{ord}_v(\wp) = 1$. If $v = \infty$, then we set $|\theta|_\infty = q$ and $\mathrm{ord}_\infty(\theta) = -\deg(\theta) = -1$. For any place v we let A_v and K_v denote the v-adic completions of A and K. For the place ∞, we note that $K_\infty = \mathbb{F}_q((1/\theta))$, and we let C_∞ be a completion of an algebraic closure of K_∞. Finally, we let $\Omega := C_\infty \setminus K_\infty$ be the Drinfeld upper half-plane of C_∞.

For $i \geqslant 1$, we set

$$[i] = \theta^{q^i} - \theta, \quad D_i = [i][i-1]^q \cdots [1]^{q^{i-1}}, \quad L_i = (-1)^i[i][i-1]\cdots[1], \tag{2.1}$$

and we let $D_0 = L_0 = 1$. We have the recursions, $D_i = [i]D_{i-1}^q$ and $L_i = -[i]L_{i-1}$, and we recall [13, Prop. 3.1.6] that

$$[i] = \prod_{\substack{f\in A_+,\ \text{irred.}\\ \deg(f)\mid i}} f, \quad D_i = \prod_{a\in A_{i+}} a, \quad L_i = (-1)^i \cdot \operatorname{lcm}(f \in A_{i+}). \tag{2.2}$$

For $m \in \mathbb{Z}_+$, we define the Carlitz factorial Π_m as follows. If we write $m = \sum m_i q^i$ with $0 \leqslant m_i \leqslant q-1$, then

$$\Pi_m = \prod_i D_i^{m_i}. \tag{2.3}$$

For more information about Π_m the reader is directed to Goss [13, §9.1].

For an $\mathbb{F}_q$-algebra L, we let $\tau : L \to L$ denote the q-th power Frobenius map, and we let $L[\tau]$ denote the ring of twisted polynomials over L, subject to the condition that $\tau c = c^q\tau$ for $c \in L$. We then define as usual the Carlitz module to be the $\mathbb{F}_q$-algebra homomorphism $C : A \to A[\tau]$ determined by

$$C_\theta = \theta + \tau.$$

The Carlitz exponential is the $\mathbb{F}_q$-linear power series,

$$e_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}. \tag{2.4}$$

The induced function $e_C : C_\infty \to C_\infty$ is both entire and surjective, and for all $a \in A$,

$$e_C(az) = C_a(e_C(z)).$$

The kernel Λ_C of $e_C(z)$ is the A-lattice of rank 1 given by $\Lambda_C = A\widetilde{\pi}$, where for a fixed $(q-1)$-st root of $-\theta$,

$$\widetilde{\pi} = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \Bigl(1 - \theta^{1-q^i}\Bigr)^{-1} \in K_\infty\bigl((-\theta)^{1/(q-1)}\bigr)$$

is called the Carlitz period (see [13, §3.2] or [18, §3.1]). Moreover, we have a product expansion

$$e_C(z) = z \prod_{\lambda\in\Lambda_C}{}' \left(1 - \frac{z}{\lambda}\right) = z \prod_{a\in A}{}' \left(1 - \frac{z}{a\widetilde{\pi}}\right), \tag{2.5}$$

where the prime indicates omitting the $a = 0$ term in the product. For more information about the Carlitz module, and Drinfeld modules in general, we refer the reader to [13, Chs. 3–4].

We will say that a function $f : \Omega \to C_\infty$ is holomorphic if it is rigid analytic in the sense of [6]. We set $\mathcal{H}(\Omega)$ to be the set of holomorphic functions on Ω. We define a holomorphic function $u : \Omega \to C_\infty$ by setting

$$u(z) := \frac{1}{e_C(\widetilde{\pi}z)}, \tag{2.6}$$

and we note that $u(z)$ is a uniformizing parameter at the infinite cusp of Ω (see [7, §5]), which plays the role of $\mathbf{q}(z) = e^{2\pi i z}$ in the classical case. The function $u(z)$ is A-periodic in the sense that $u(z+a) = u(z)$ for all $a \in A$. The imaginary part of an element $z \in C_\infty$ is set to be

$$|z|_i = \inf_{x \in K_\infty} |z - x|_\infty,$$

which measures the distance from z to the real axis $K_\infty \subseteq C_\infty$. We will say that an A-periodic holomorphic function $f : \Omega \to C_\infty$ is holomorphic at ∞ if we can write a convergent series,

$$f(z) = \sum_{n=0}^{\infty} c_n u(z)^n, \quad c_n \in C_\infty, \quad |z|_i \gg 0.$$

The function f is then determined by the power series $f = \sum c_n u^n \in C_\infty[[u]]$, and we call this power series the u-expansion of f and the coefficients c_n the u-expansion coefficients of f. We set $\mathcal{U}(\Omega)$ to be the subset of $\mathcal{H}(\Omega)$ comprising functions on Ω that are A-periodic and holomorphic at ∞. In other words, $\mathcal{U}(\Omega)$ consists of functions that have u-expansions.

We now define hyperdifferential operators and hyperderivatives (see [5], [15], [22] for more details). For a field F and an independent variable z over F, for $j \geqslant 0$ we define the j-th hyperdifferential operator $\partial_z^j : F[z] \to F[z]$ by setting

$$\partial_z^j(z^n) = \binom{n}{j} z^{n-j}, \quad n \geqslant 0,$$

where $\binom{n}{j} \in \mathbb{Z}$ is the usual binomial coefficient, and extending F-linearly. (By usual convention $\binom{n}{j} = 0$ if $0 \leqslant n < j$.) For $f \in F[z]$, we call $\partial_z^j(f) \in F[z]$ its j-th hyperderivative. Hyperderivatives satisfy the product rule,

$$\partial_z^j(fg) = \sum_{k=0}^{j} \partial_z^k(f)\partial_z^{j-k}(g), \quad f, g \in F[z], \tag{2.7}$$

and composition rule,

$$(\partial_z^j \circ \partial_z^k)(f) = (\partial_z^k \circ \partial_z^j)(f) = \binom{j+k}{j} \partial_z^{j+k}(f), \quad f \in F[z]. \tag{2.8}$$

Using the product rule one can extend to $\partial_z^j : F(z) \to F(z)$ in a unique way, and $F(z)$ together with the operators ∂_z^j form a hyperdifferential system. If F has characteristic 0, then $\partial_z^j = \frac{1}{j!}\frac{d^j}{dz^j}$, but in characteristic p this holds only for $j \leqslant p-1$. Furthermore, hyperderivatives satisfy a number of differentiation rules (e.g., product, quotient, power, chain rules), which aid in their description and calculation (see [15, §2.2], [18, §2.3], for a complete list of rules and historical accounts). Moreover, if $f \in F(z)$ is regular at $c \in F$, then so is $\partial_z^j(f)$ for each $j \geqslant 0$, and it follows that we have a Taylor expansion,

$$f(z) = \sum_{j=0}^{\infty} \partial_z^j(f)(c) \cdot (z-c)^j \in F[[z-c]]. \tag{2.9}$$

In this way we can also extend ∂_z^j uniquely to $\partial_z^j : F((z-c)) \to F((z-c))$.

For a holomorphic function $f : \Omega \to C_\infty$, it was proved by Uchino and Satoh [22, §2] that we can define a holomorphic hyperderivative $\partial_z^j(f) : \Omega \to C_\infty$ (taking $F = C_\infty$ in the preceding paragraph). That is,

$$\partial_z^j : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega).$$

Moreover they prove that the system of operators ∂_z^j on holomorphic functions inherits the same differentiation rules for hyperderivatives of polynomials and power series. Thus for $f \in \mathcal{H}(\Omega)$ and $c \in \Omega$, we have a Taylor expansion,

$$f(z) = \sum_{j=0}^{\infty} \partial_z^j(f)(c) \cdot (z-c)^j \in C_\infty[[z-c]].$$

We have the following crucial lemma for our later considerations in §4, where we find new identities for derivatives of functions in $\mathcal{U}(\Omega)$.

Lemma 2.10 (Uchino-Satoh [22, Lem. 3.6]). *If $f \in \mathcal{H}(\Omega)$ is A-periodic and holomorphic at ∞, then so is $\partial_z^j(f)$ for each $j \geqslant 0$. That is,*

$$\partial_z^j : \mathcal{U}(\Omega) \to \mathcal{U}(\Omega), \quad j \geqslant 0.$$

We recall computations involving $u(z)$ and $\partial_z^1(u(z))$ (see [7, §3]). First we see from (2.4) that $\partial_z^1(e_C(z)) = 1$, so using (2.5) and taking logarithmic derivatives,

$$u(z) = \frac{1}{e_C(\widetilde{\pi} z)} = \frac{1}{\widetilde{\pi}} \cdot \frac{\partial_z^1(e_C(\widetilde{\pi} z))}{e_C(\widetilde{\pi} z)} = \frac{1}{\widetilde{\pi}} \sum_{a \in A} \frac{1}{z+a}. \tag{2.11}$$

Furthermore,

$$\partial_z^1(u(z)) = \partial_z^1\left(\frac{1}{e_C(\widetilde{\pi} z)}\right) = \frac{-\partial_z^1(e_C(\widetilde{\pi} z))}{e_C(\widetilde{\pi} z)^2} = -\widetilde{\pi} u(z)^2. \tag{2.12}$$

Thus, $\partial_z^1(u) = -\widetilde{\pi} u^2 \in \mathcal{U}(\Omega)$. In §4 we generalize this formula and calculate $\partial_z^r(u^n)$ for $r, n \geqslant 0$.

We conclude this section by discussing some properties of hyperderivatives particular to positive characteristic. Suppose $\mathrm{char}(F) = p > 0$. If we write $j = \sum_{i=0}^s b_i p^i$, with $0 \leqslant b_i \leqslant p-1$ and $b_s \neq 0$, then (see [15, Thm. 3.1])

$$\partial_z^j = \partial_z^{b_0} \circ \partial_z^{b_1 p} \circ \cdots \circ \partial_z^{b_s p^s}, \tag{2.13}$$

which follows from the composition law and Lucas's theorem (e.g., see [1, Eq. (14)]). We note that for $0 \leqslant b \leqslant p-1$,

$$\partial_z^{bp^k} = \frac{1}{b!} \cdot \partial_z^{p^k} \circ \cdots \circ \partial_z^{p^k}, \quad (b \text{ times}).$$

Moreover the p-th power rule (see [3, §7], [15, §2.2]) says that for $f \in F((z-c))$,

$$\partial_z^j\big(f^{p^s}\big) = \begin{cases} \big(\partial_z^\ell(f)\big)^{p^s} & \text{if } j = \ell p^s, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

and so calculation using (2.13) and (2.14) can often be fairly efficient.

3. Goss polynomials and hyperderivatives

We review here results on Goss polynomials, which were introduced by Goss in [12, §6] and have been studied further by Gekeler [7, §3], [8]. We start first with an $\mathbb{F}_q$-vector space $\Lambda \subseteq C_\infty$ of dimension d. We define the exponential function of Λ,

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{z}{\lambda}\right),$$

which is an $\mathbb{F}_q$-linear polynomial of degree q^d. If we take $t_\Lambda(z) = 1/e_\Lambda(z)$, then just as in (2.11) we have

$$t_\Lambda(z) = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}.$$

We can extend these definitions to any discrete lattice $\Lambda \subseteq C_\infty$, which is the union of nested finite dimensional $\mathbb{F}_q$-vector spaces $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots$. We find that generally $e_\Lambda(z) = \lim_{i\to\infty} e_{\Lambda_i}(z)$ and $t_\Lambda(z) = \lim_{i\to\infty} t_{\Lambda_i}(z)$, where the convergence is coefficient-wise in $C_\infty((z))$.

Remark 3.1. If we take $\Lambda = \Lambda_C$, then $e_{\Lambda_C}(z) = e_C(z)$, whereas if we take $\Lambda = A$, then $e_A(z) = \frac{1}{\widetilde{\pi}} e_C(\widetilde{\pi} z)$. Thus

$$t_A(z) = \widetilde{\pi}\, t_{\Lambda_C}(\widetilde{\pi} z) = \frac{\widetilde{\pi}}{e_C(\widetilde{\pi} z)},$$

and $u(z)$, as defined in (2.6), is given by

$$u(z) = \frac{t_A(z)}{\widetilde{\pi}} = t_{\Lambda_C}(\widetilde{\pi} z).$$

This normalization of $u(z)$ is taken so that the u-expansions of some Drinfeld modular forms will have K-rational coefficients.

Theorem 3.2 (Goss [12, §6]; see also Gekeler [7, §3])**.** *Let $\Lambda \subseteq C_\infty$ be a discrete $\mathbb{F}_q$-vector space. Let*

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{z}{\lambda}\right) = \sum_{j=0}^{\infty} \alpha_j z^{q^j},$$

and let $t_\Lambda(z) = 1/e_\Lambda(z)$. For each $k \geqslant 1$, there is a monic polynomial $G_{k,\Lambda}(t)$ of degree k with coefficients in $\mathbb{F}_q[\alpha_0, \alpha_1, \ldots]$ so that

$$S_{k,\Lambda}(z) := \sum_{\lambda \in \Lambda} \frac{1}{(z - \lambda)^k} = G_{k,\Lambda}\big(t_\Lambda(z)\big).$$

Furthermore the following properties hold.

(a) $G_{k,\Lambda}(t) = t(G_{k-1,\Lambda}(t) + \alpha_1 G_{k-q,\Lambda}(t) + \alpha_2 G_{k-q^2,\Lambda}(t) + \cdots)$.

(b) *We have a generating series identity*

$$\mathcal{G}_\Lambda(t, x) = \sum_{k=1}^{\infty} G_{k,\Lambda}(t) x^k = \frac{tx}{1 - t e_\Lambda(x)}.$$

(c) *If $k \leqslant q$, then $G_{k,\Lambda}(t) = t^k$.*

(d) $G_{pk,\Lambda}(t) = G_{k,\Lambda}(t)^p$.

(e) $t^2\, \partial_t^1\big(G_{k,\Lambda}(t)\big) = k G_{k+1,\Lambda}(t)$.

Gekeler [7, (3.8)] finds a formula for each $G_{k,\Lambda}(t)$,

$$G_{k+1,\Lambda}(t) = \sum_{j=0}^{k} \sum_{\underline{i}} \binom{j}{\underline{i}} \alpha^{\underline{i}} t^{j+1}, \tag{3.3}$$

where the sum is over all $(s+1)$-tuples $\underline{i} = (i_0, \ldots, i_s)$, with s arbitrary, satisfying $i_0 + \cdots + i_s = j$ and $i_0 + i_1 q + \cdots + i_s q^s = k$; $\binom{j}{\underline{i}} = j!/(i_0! \cdots i_s!)$ is a multinomial coefficient; and $\alpha^{\underline{i}} = \alpha_0^{i_0} \cdots \alpha_s^{i_s}$.

Part (e) of Theorem 3.2 indicates that there are interesting hyperderivative relations among Goss polynomials, with respect to t and to z, which we now investigate. All hyperderivatives we will take will be of polynomials and formal power series, but the considerations in §2 about holomorphic functions will play out later in the paper. The main result of this section is the following.

Theorem 3.4. *Let $\Lambda \subseteq C_\infty$ be a discrete $\mathbb{F}_q$-vector space, and let $t = t_\Lambda(z)$. For $r \geqslant 0$, we define $\beta_{r,j}$ so that*

$$G_{r+1,\Lambda}(t) = \sum_{j=0}^{r} \beta_{r,j} t^{j+1}.$$

Then

$$\partial_z^r(t^n) = (-1)^r \cdot t^n \, \partial_t^{n-1}\bigl(t^{n-2} G_{r+1,\Lambda}(t)\bigr), \quad \forall\, n \geqslant 1, \tag{3.4a}$$

$$\partial_z^r(t^n) = (-1)^r \sum_{j=0}^{r} \beta_{r,j}\, t^{j+1}\, \partial_t^j(t^{n+j-1}), \quad \forall\, n \geqslant 0, \tag{3.4b}$$

and

$$\binom{n+r-1}{r} G_{n+r,\Lambda}(t) = \sum_{j=0}^{r} \beta_{r,j}\, t^{j+1}\, \partial_t^j\bigl(t^{j-1} G_{n,\Lambda}(t)\bigr), \quad \forall\, n \geqslant 1. \tag{3.4c}$$

Remark 3.5. We see that (3.4a) and (3.4b) generalize (2.12) and that (3.4c) generalizes Theorem 3.2(e). In later sections (3.4a) and (3.4b) will be useful for taking derivatives of Drinfeld modular forms. The coefficients $\beta_{r,j}$ can be computed using the generating series $\mathcal{G}_\Lambda(t,x)$ or equivalently (3.3). The proof requires some preliminary lemmas.

Lemma 3.6 (cf. Petrov [20, §3])**.** *For $r \geqslant 0$ and $n \geqslant 1$,*

$$\partial_z^r\bigl(S_{n,\Lambda}(z)\bigr) = (-1)^r \binom{n+r-1}{r} G_{n+r,\Lambda}(t). \tag{3.6a}$$

Moreover, we have

$$\partial_z^r\bigl(S_{n,\Lambda}(z)\bigr) = (-1)^{n+r-1} \cdot \partial_z^{n-1}\bigl(S_{r+1,\Lambda}(z)\bigr) \tag{3.6b}$$

and

$$\partial_z^r(t) = (-1)^r G_{r+1,\Lambda}(t). \tag{3.6c}$$

Proof. First of all, we recall the convention that for $n > 0$ and $r \geqslant 0$, we have $\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}$. Then using the power and quotient rules [15, §2.2], we see that for $\lambda \in C_\infty$,

$$\partial_z^r\left(\frac{1}{(z-\lambda)^n}\right) = \binom{-n}{r}\frac{1}{(z-\lambda)^{n+r}} = (-1)^r\binom{n+r-1}{r}\frac{1}{(z-\lambda)^{n+r}}.$$

Therefore,

$$\partial_z^r\bigl(S_{n,\Lambda}(z)\bigr) = (-1)^r\binom{n+r-1}{r} S_{n+r,\Lambda}(z),$$

and combining with the defining property of $G_{n+r,\Lambda}(t)$ in Theorem 3.2, we see that (3.6a) follows. Now

$$\binom{n+r-1}{r} = \binom{(r+1)+(n-1)-1}{n-1},$$

and so (3.6b) follows from (3.6a). Finally, (3.6c) is a special case of (3.6a) with $n = 1$. $\square$

Lemma 3.7. *For $n \geqslant 1$, we have an identity of rational functions in x,*

$$\frac{x}{(1 - te_\Lambda(x))^n} = \partial_t^{n-1}\left(\frac{t^{n-1}x}{1 - te_\Lambda(x)}\right) = \partial_t^{n-1}\bigl(t^{n-2}\mathcal{G}_\Lambda(t,x)\bigr).$$

Proof. Our derivatives with respect to t are taken while considering x to be a constant. We note that for $\ell \geqslant 0$,

$$\partial_t^\ell\left(\frac{1}{1 - te_\Lambda(x)}\right) = \frac{e_\Lambda(x)^\ell}{(1 - te_\Lambda(x))^{\ell+1}},$$

by the quotient and chain rules [15, §2.2]. Therefore, by the product rule,

$$\begin{aligned}
\partial_t^{n-1}\left(\frac{t^{n-1}}{1 - te_\Lambda(x)}\right) &= \sum_{k=0}^{n-1} \partial_t^k(t^{n-1})\partial_t^{n-1-k}\left(\frac{1}{1 - te_\Lambda(x)}\right) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k}\left(\frac{te_\Lambda(x)}{1 - te_\Lambda(x)}\right)^{n-1-k} \cdot \frac{1}{1 - te_\Lambda(x)} \\
&= \left(1 + \frac{te_\Lambda(x)}{1 - te_\Lambda(x)}\right)^{n-1} \cdot \frac{1}{1 - te_\Lambda(x)}.
\end{aligned}$$

A simple calculation yields that this is $1/(1 - te_\Lambda(x))^n$, and the result follows. $\square$

Proof of Theorem 3.4. The chain rule [15, §2.2] and (3.6c) imply that

$$\begin{aligned}
\partial_z^r(t^n) &= \sum_{k=1}^{r} \binom{n}{k} t^{n-k} \sum_{\substack{\ell_1,\ldots,\ell_k \geqslant 1 \\ \ell_1+\cdots+\ell_k = r}} \partial_z^{\ell_1}(t) \cdots \partial_z^{\ell_k}(t) \\
&= (-1)^r \sum_{k=1}^{r} \binom{n}{k} t^{n-k} \sum_{\substack{\ell_1,\ldots,\ell_k \geqslant 1 \\ \ell_1+\cdots+\ell_k = r}} G_{\ell_1+1,\Lambda}(t) \cdots G_{\ell_k+1,\Lambda}(t).
\end{aligned}$$

By direct expansion (see [15, §2.2, Eq. (I)]), the final inner sum above is the coefficient of x^r in

$$\bigl(G_{2,\Lambda}(t)x + G_{3,\Lambda}(t)x^2 + \cdots\bigr)^k,$$

and therefore by the binomial theorem,

$$\partial_z^r(t^n) = (-1)^r \cdot \Bigl(\text{coefficient of } x^r \text{ in } \bigl(t + G_{2,\Lambda}(t)x + G_{3,\Lambda}(t)x^2 + \cdots\bigr)^n\Bigr).$$

Now $G_{1,\Lambda}(t) = t$, so

$$t + G_{2,\Lambda}(t)x + G_{3,\Lambda}(t)x^2 + \cdots = \sum_{k=1}^{\infty} G_{k,\Lambda}(t)x^{k-1} = \frac{\mathcal{G}_\Lambda(t,x)}{x} = \frac{t}{1 - te_\Lambda(x)}.$$

Therefore,

$$\partial_z^r(t^n) = (-1)^r \cdot \left(\text{coefficient of } x^{r+1} \text{ in } \frac{t^n x}{(1 - te_\Lambda(x))^n}\right).$$

From Lemma 3.7 we see that
$$\frac{t^n x}{(1-te_\Lambda(x))^n} = t^n \sum_{k=1}^{\infty} \partial_t^{n-1}\bigl(t^{n-2}G_{k,\Lambda}(t)\bigr)x^k,$$
and so (3.4a) holds. To prove (3.4b), we first note that it holds when $n=0$ by checking the various cases and using that $\beta_{r,0}=0$ for $r \geqslant 1$, since $G_{r+1,\Lambda}(t)$ is divisible by t^2 for $r \geqslant 1$ by Theorem 3.2, and that $\beta_{0,0}=1$. For $n \geqslant 1$, we use (3.4a) and write
$$\partial_z^r(t^n) = (-1)^r \cdot t^n\, \partial_t^{n-1}\bigl(t^{n-2}G_{r+1,\Lambda}(t)\bigr) = (-1)^r \cdot t^n\, \partial_t^{n-1}\Biggl(\sum_{j=0}^{r} \beta_{r,j}\, t^{n+j-1}\Biggr).$$
Noting that
$$\partial_t^{n-1}(t^{n+j-1}) = \binom{n+j-1}{n-1} t^j = t^{j-n+1}\, \partial_t^j(t^{n+j-1}),$$
we then have
$$\partial_z^r(t^n) = (-1)^r \sum_{j=0}^{r} \beta_{r,j}\, t^{j+1}\, \partial_t^j(t^{n+j-1}),$$
and so (3.4b) holds. Furthermore, by (3.6a) and (3.6b),
$$\binom{n+r-1}{r} G_{n+r,\Lambda}(t) = (-1)^{n-1} \cdot \partial_z^{n-1}\bigl(S_{r+1,\Lambda}(z)\bigr) = (-1)^{n-1} \cdot \partial_z^{n-1}\bigl(G_{r+1,\Lambda}(t)\bigr).$$
But then by (3.4a),
$$\partial_z^{n-1}\bigl(G_{r+1,\Lambda}(t)\bigr) = \sum_{j=0}^{r} \beta_{r,j} \partial_z^{n-1}(t^{j+1}) = (-1)^{n-1} \sum_{j=0}^{r} \beta_{r,j} t^{j+1}\, \partial_t^j\bigl(t^{j-1}G_{n,\Lambda}(t)\bigr),$$
which yields (3.4c). □

4. Theta operators on Drinfeld modular forms

We recall the definition of Drinfeld modular forms for $\mathrm{GL}_2(A)$, which were initially studied by Goss [10], [11], [12]. We will also review results on u-expansions of modular forms due to Gekeler [7]. Throughout we let $\Gamma = \mathrm{GL}_2(A)$. A holomorphic function $f : \Omega \to C_\infty$ is a Drinfeld modular form of weight $k \geqslant 0$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ if

(1) for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \Omega$,
$$f(\gamma z) = (\det \gamma)^{-m}(cz+d)^k f(z), \quad \gamma z = \frac{az+b}{cz+d};$$

(2) and f is holomorphic at ∞, i.e., f has a u-expansion and so $f \in \mathcal{U}(\Omega)$.

We let M_k^m be the C_∞-vector space of modular forms of weight k and type m. We know that $M_k^m \cdot M_{k'}^{m'} \subseteq M_{k+k'}^{m+m'}$ and that $M = \bigoplus_{k,m} M_k^m$ and $M^0 = \bigoplus_k M_k^0$ are graded C_∞-algebras. Moreover, in order to have $M_k^m \neq 0$, we must have $k \equiv 2m \pmod{q-1}$. If L is a subring of C_∞, then we let $M_k^m(L)$ denote the space of forms with u-expansion coefficients in L, i.e., $M_k^m(L) = M_k^m \cap L[[u]]$. We note that if $f = \sum c_n u^n$ is the u-expansion of $f \in M_k^m$, then

(4.1) $$c_n \neq 0 \quad \Rightarrow \quad n \equiv m \pmod{q-1},$$

which can be seen by using $\gamma = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for ζ a generator of $\mathbb{F}_q^\times$, in the definition above.

Certain Drinfeld modular forms can be expressed in terms of A-expansions, which we now recall. For $k \geqslant 1$, we set

$$G_k(t) = G_{k,\Lambda_C}(t) = \sum_{j=0}^{k-1} \beta_{k-1,j} t^{j+1}, \tag{4.2}$$

to be the Goss polynomials with respect to the lattice Λ_C. Since $e_C(z) \in K[[z]]$, it follows from Theorem 3.2 that the coefficients $\beta_{k-1,j} \in K$ for all k, j. As in (2.6) and Remark 3.1, we have $u(z) = 1/e_C(\widetilde{\pi} z)$, and for $a \in A$ we set

$$u_a(z) := u(az) = \frac{1}{e_C(\widetilde{\pi} az)}. \tag{4.3}$$

Since $e_C(\widetilde{\pi} az) = C_a(e_C(\widetilde{\pi} z))$, if we take the reciprocal polynomial for $C_a(z)$ to be $R_a(z) = z^{q^{\deg a}} C_a(1/z)$ then

$$u_a = \frac{u^{q^{\deg a}}}{R_a(u)} = u^{q^{\deg a}} + \cdots \in A[[u]]. \tag{4.4}$$

We say that a modular form f has an A-expansion if there exist $k \geqslant 1$ and c_0, $c_a \in C_\infty$ for $a \in A_+$, so that

$$f = c_0 + \sum_{a \in A_+} c_a G_k(u_a).$$

Example 4.5. For $k \equiv 0 \pmod{q-1}$, $k > 0$, the primary examples of Drinfeld modular forms with A-expansions come from Eisenstein series,

$$E_k(z) = \frac{1}{\widetilde{\pi}^k} \sideset{}{'}\sum_{a,b \in A} \frac{1}{(az+b)^k},$$

which is a modular form of weight k and type 0. Gekeler [7, (6.3)] showed that

$$E_k = \frac{1}{\widetilde{\pi}^k} \sideset{}{'}\sum_{b \in A} \frac{1}{b^k} - \sum_{a \in A_+} G_k(u_a) = -\frac{\zeta_C(k)}{\widetilde{\pi}^k} - \sum_{a \in A_+} G_k(u_a), \tag{4.6}$$

where $\zeta_C(k) = \sum_{a \in A_+} a^{-k}$ is a Carlitz zeta value. We know (see [13, §9.2]) that $\zeta_C(k)/\widetilde{\pi}^k \in K$.

For more information and examples on A-expansions the reader is directed to Gekeler [7], López [16], [17], and Petrov [19], [20].

Example 4.7. We can also define the false Eisenstein series $E(z)$ of Gekeler [7, §8] to be

$$E(z) := \frac{1}{\widetilde{\pi}} \sum_{a \in A_+} \sum_{b \in A} \frac{a}{az+b},$$

which is not quite a modular form but is a quasi-modular form similar to the classical weight 2 Eisenstein series [1], [7]. Gekeler showed that $E \in \mathcal{U}(\Omega)$ and that E has an A-expansion,

$$E = \sum_{a \in A_+} a G_1(u_a) = \sum_{a \in A_+} a u_a. \tag{4.8}$$

We now define theta operators Θ^r on functions in $\mathcal{H}(\Omega)$ by setting for $r \geqslant 0$,

$$\Theta^r := \frac{1}{(-\widetilde{\pi})^r} \partial_z^r. \tag{4.9}$$

If we take $\Theta = \Theta^1$, then by (2.12), $\Theta u = u^2$, and Θ plays the role of the classical theta operator $\vartheta = \mathbf{q}\frac{d}{d\mathbf{q}}$. Just as in the classical case, Θ and more generally Θ^r do not take modular forms to modular forms. However, Bosser and Pellarin [1, Thm. 2] prove that Θ^r preserves quasi-modularity:

$$\Theta^r : C_\infty[E, g, h] \to C_\infty[E, g, h],$$

where E is the false Eisenstein series, $g = E_{q-1}$, and h is the cusp form of weight $q+1$ and type 1 defined by Gekeler [7, Thm. 5.13] as the $(q-1)$-st root of the discriminant function Δ. To prove their theorem, Bosser and Pellarin [1, Lem. 3.5] give formulas for $\Theta^r(u^n)$, which are ostensibly a bit complicated. From Theorem 3.4, we have the following corollary, which perhaps conceptually simplifies matters.

Corollary 4.10. *For $r \geqslant 0$,*

$$\Theta^r(u^n) = u^n\, \partial_u^{n-1}\bigl(u^{n-2} G_{r+1}(u)\bigr), \quad \forall\, n \geqslant 1, \tag{4.10a}$$

$$\Theta^r(u^n) = \sum_{j=0}^{r} \beta_{r,j} u^{j+1} \partial_u^j (u^{n+j-1}) = \sum_{j=0}^{r} \binom{n+j-1}{j} \beta_{r,j} u^{n+j}, \quad \forall\, n \geqslant 0, \tag{4.10b}$$

where $\beta_{r,j}$ are the coefficients of $G_{r+1}(t)$ in (4.2).

Proof. The proof of (4.10a) is straightforward, but it is worth noting how the different normalizations of $u(z)$ and $t_{\Lambda_C}(z)$ work out. From Remark 3.1, we see that

$$\begin{aligned}\Theta^r(u^n) = \left(\frac{-1}{\widetilde{\pi}}\right)^r \partial_z^r\bigl(t_{\Lambda_C}(\widetilde{\pi} z)^n\bigr) &= \left(\frac{-1}{\widetilde{\pi}}\right)^r \cdot \widetilde{\pi}^r\, \partial_z^r\bigl(t_{\Lambda_C}^n\bigr)\big|_{z=\widetilde{\pi} z} \\ &= t^n\, \partial_t^{n-1}\bigl(t^{n-2} G_{r+1}(t)\bigr)\big|_{t=t_{\Lambda_C}(\widetilde{\pi} z)} = u^n\, \partial_u^{n-1}\bigl(u^{n-2} G_{r+1}(u)\bigr),\end{aligned}$$

where the third equality is (3.4a). The proof of (4.10b) is then the same as for (3.4b). □

Remark 4.11. We see from (4.10a) that there is a duality of some fashion between the r-th derivative of u^n and the $(n-1)$-st derivative of $G_{r+1}(u)$, which dovetails with (3.6b).

We see from this corollary that Θ^r can be seen as the operator on power series in $C_\infty[[u]]$ given by the following result. Moreover, from (4.12b), we see that computation of $\Theta^r(f)$ is reasonably straightforward once the computation of the coefficients of $G_{r+1}(t)$ can be made.

Corollary 4.12. *Let $f = \sum c_n u^n \in \mathcal{U}(\Omega)$. For $r \geqslant 0$,*

$$\Theta^r(f) = \Theta^r(c_0) + \sum_{n=1}^{\infty} c_n u^n\, \partial_u^{n-1}\bigl(u^{n-2} G_{r+1}(u)\bigr), \tag{4.12a}$$

$$\Theta^r(f) = \sum_{j=0}^{r} \beta_{r,j}\, u^{j+1}\, \partial_u^j\bigl(u^{j-1} f\bigr), \tag{4.12b}$$

where $\beta_{r,j}$ are the coefficients of $G_{r+1}(t)$ in (4.2).

Finally we recall the definition of the r-th Serre operator $\mathcal{D}^r$ on modular forms in M_k^m for $r \geqslant 0$. We set

$$\mathcal{D}^r(f) := \Theta^r(f) + \sum_{i=1}^{r} (-1)^i \binom{k+r-1}{i} \Theta^{r-i}(f)\Theta^{i-1}(E). \tag{4.13}$$

The following result shows that $\mathcal{D}^r$ takes modular forms to modular forms.

Theorem 4.14 (Bosser-Pellarin [2, Thm. 4.1])**.** *For any weight k, type m, and $r \geqslant 0$,*

$$\mathcal{D}^r\left(M_k^m\right) \subseteq M_{k+2r}^{m+r}.$$

5. v-adic modular forms

In this section we review the theory of v-adic modular forms introduced by Goss [14] and Vincent [24]. In [21], Serre defined p-adic modular forms as p-adic limits of Fourier series of classical modular forms and determined their properties, in particular their behavior under the ϑ-operator. For a fixed finite place v of K, Goss and Vincent recently transferred Serre's definition to the function field setting of v-adic modular forms, and Goss produced families of examples based on work of Petrov [19] (see Theorem 6.3). In §6, we show that v-adic modular forms are invariant under the operators Θ^r.

For our place v of K we fix $\wp \in A_+$, which is the monic irreducible generator of the ideal $\mathfrak{p}_v$ associated to v, and we let $d := \deg(\wp)$. As before we let A_v and K_v denote completions with respect to v.

We will write $K \otimes A_v[[u]]$ for $K \otimes_A A_v[[u]]$, and we recall that $K \otimes A_v[[u]]$ can be identified with elements of $K_v[[u]]$ that have bounded denominators. For $f = \sum_{n=0}^{\infty} c_n u^n \in K \otimes A_v[[u]]$, we set

$$\mathrm{ord}_v(f) := \inf_n \{\mathrm{ord}_v(c_n)\} = \min_n \{\mathrm{ord}_v(c_n)\}. \tag{5.1}$$

If $\mathrm{ord}_v(f) \geqslant 0$, i.e., if $f \in A_v[[u]]$, then we say f is v-integral. For f, $g \in K \otimes A_v[[u]]$, we write that

$$f \equiv g \pmod{\wp^m},$$

if $\mathrm{ord}_v(f-g) \geqslant m$. We also define a topology on $K \otimes A_v[[u]]$ in terms of the v-adic norm,

$$\|f\|_v := q^{-\mathrm{ord}_v(f)}, \tag{5.2}$$

which is a multiplicative norm by Gauss' lemma.

Following Goss, we define the v-adic weight space $\mathbb{S} = \mathbb{S}_v$ by

$$\mathbb{S} := \varprojlim_{\ell} \mathbb{Z}/(q^d-1)p^{\ell}\mathbb{Z} = \mathbb{Z}/(q^d-1)\mathbb{Z} \times \mathbb{Z}_p. \tag{5.3}$$

We have a canonical embedding of $\mathbb{Z} \hookrightarrow \mathbb{S}$, by identifying $n \in \mathbb{Z}$ with $(\overline{n}, n)$, where $\overline{n}$ is the class of n modulo q^d-1. For any $a \in A_+$ with $\wp \nmid a$, we can decompose a as $a = a_1 a_2$, where $a_1 \in A_v^{\times}$ is the (q^d-1)-st root of unity satisfying $a_1 \equiv a \pmod{v}$ and $a_2 \in A_v^{\times}$ satisfies $a_2 \equiv 1 \pmod{v}$. Then for any $s = (x, y) \in \mathbb{S}$, we define

$$a^s := a_1^x a_2^y. \tag{5.4}$$

This definition of a^s is compatible with the usual definition when s is an integer. Furthermore, it is easy to check that the function $s \mapsto a^s$ is continuous from $\mathbb{S}$ to $A_v^{\times}$.

Definition 5.5 (Goss [14, Def. 5])**.** We say a power series $f \in K \otimes A_v[[u]]$ is a v-adic modular form of weight $s \in \mathbb{S}$, in the sense of Serre, if there exists a sequence of K-rational modular forms $f_i \in M_{k_i}^{m_i}(K)$ so that as $i \to \infty$,

(a) $\|f_i - f\|_v \to 0$,
(b) $k_i \to s$ in $\mathbb{S}$.

Moreover, if $f \neq 0$, then m_i is eventually a constant $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, and we say that m is the type of f. We say that f_i converges to f as v-adic modular forms.

It is easy to see that the sum and difference of two v-adic modular forms, both with weight s and type m, are also v-adic modular forms with the same weight and type. We set

$$\mathcal{M}_s^m = \bigl\{ f \in K \otimes A_v[[u]] \bigm| f \text{ a } v\text{-adic modular form of weight } s \text{ and type } m \bigr\}, \tag{5.6}$$

which is a K_v-vector space, and we note that

$$\mathcal{M}_{s_1}^{m_1} \cdot \mathcal{M}_{s_2}^{m_2} \subseteq \mathcal{M}_{s_1+s_2}^{m_1+m_2}.$$

We take $\mathcal{M}_s^m(A_v) := \mathcal{M}_s^m \cap A_v[[u]]$, which is an A_v-module. Moreover, any Drinfeld modular form in $M_k^m(K)$ is also a v-adic modular form as the limit of the constant sequence (u-expansion coefficients of forms in $M_k^m(K)$ have bounded denominators by [7, Thm. 5.13, §12], [11, Thm. 2.23]), and so for $k \in \mathbb{Z}$, $k \geqslant 0$,

$$M_k^m(K) \subseteq \mathcal{M}_k^m, \quad M_k^m(A) \subseteq \mathcal{M}_k^m(A_v).$$

The justification of the final part of Definition 5.5 is the following lemma.

Lemma 5.7. *Suppose that $f_i \in M_{k_i}^{m_i}(K)$ converge to a non-zero v-adic modular form f. Then there is some $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ so that except for finitely terms $m_i = m$.*

Proof. Since $\|f - f_i\|_v \to 0$, it follows that $\|f_i - f_j\|_v \to 0$ as $i, j \to \infty$. If $f = \sum c_n u^n$ and $c_n \neq 0$, then from (4.1) we see that for $i, j \gg 0$, $n \equiv m_i \equiv m_j \pmod{q-1}$. □

Proposition 5.8. *Suppose $\{f_i\}$ is a sequence of v-adic modular forms with weights s_i. Suppose that we have $f_0 \in K \otimes A_v[[u]]$ and $s_0 \in \mathbb{S}$ satisfying,*

(a) $\|f_i - f_0\|_v \to 0$,
(b) $s_i \to s_0$ in $\mathbb{S}$.

Then f_0 is a v-adic modular form of weight s_0. The type of f_0 is the eventual constant type of the sequence $\{f_i\}$.

Proof. For each $i \geqslant 1$, we have a sequence of Drinfeld modular forms $g_{i,j} \to f_i$ as $j \to \infty$. Standard arguments show that the sequence of Drinfeld modular forms $\{g_{i,i}\}_{i=1}^{\infty}$ converges to f_0 with respect to the $\|\cdot\|_v$-norm and that the weights k_i of $g_{i,i}$ go to s_0 in $\mathbb{S}$. □

We recall the definitions of Hecke operators on Drinfeld modular forms and their actions on u-expansions [7, §7], [12, §7]. For $\ell \in A_+$ irreducible of degree e, the Hecke operator $T_\ell : M_k^m \to M_k^m$ is defined by

$$(T_\ell f)(z) = \ell^k f(\ell z) + U_\ell f(z) = \ell^k f(\ell z) + \sum_{\beta \in A_{(<e)}} f\biggl(\frac{z+\beta}{\ell}\biggr).$$

Just as in the classical case the operators T_ℓ and U_ℓ are uniquely determined by their actions on u-expansions. We define U_ℓ, $V_\ell : C_\infty[[u]] \to C_\infty[[u]]$ by

$$U_\ell\Bigg(\sum_{n=0}^{\infty} c_n u^n\Bigg) := \sum_{n=1}^{\infty} c_n G_{n,\Lambda_\ell}(\ell u), \tag{5.9}$$

where $\Lambda_\ell \subseteq C_\infty$ is the e-dimensional $\mathbb{F}_q$-vector space of ℓ-division points on the Carlitz module C, and

$$V_\ell\Bigg(\sum_{n=0}^{\infty} c_n u^n\Bigg) := \sum_{n=0}^{\infty} c_n u_\ell^n. \tag{5.10}$$

We find [7, Eq. (7.3)] that $T_\ell : C_\infty[[u]] \to C_\infty[[u]]$ of weight k is given by $T_\ell = \ell^k V_\ell + U_\ell$.

If $f \in \mathcal{M}_s^m$ for some weight $s \in \mathbb{S}$, then we define $U_\ell(f)$, $V_\ell(f) \in K \otimes A_v[[u]]$ as above, and if $\ell \neq \wp$, we set

$$T_\ell(f) = \ell^s V_\ell(f) + U_\ell(f), \tag{5.11}$$

where ℓ^s is defined as in (5.4) (note that if $\ell = \wp$, then (5.4) is not well-defined). Of importance to us is that Hecke operators preserve spaces of v-adic modular forms.

Proposition 5.12. *Let $\ell \in A_+$ be irreducible, $\ell \neq \wp$. For all v-adic weights s and types m, the operators T_ℓ, $U_\wp$, and $V_\wp$ preserve the spaces $\mathcal{M}_s^m$ and $\mathcal{M}_s^m(A_v)$.*

We first define a sequence of normalized Eisenstein series studied by Gekeler [7, §6]. For $d \geqslant 1$, we let

$$g_d(z) = -L_d \cdot E_{q^d-1}(z), \tag{5.13}$$

which is a Drinfeld modular form of weight q^d-1 and type 0. By the following proposition we see that g_d plays the role of E_{p-1} for classical modular forms.

Proposition 5.14 (Gekeler [7, Prop. 6.9, Cor. 6.12])**.** *For $d \geqslant 1$, the following hold:*

(a) $g_d \in A[[u]]$;
(b) $g_d \equiv 1 \pmod{[d]}$.

Proof of Proposition 5.12. Let $f \in \mathcal{M}_s^m$. Once we establish that $T_\ell(f)$, $U_\wp(f)$, and $V_\wp(f)$ are elements of $\mathcal{M}_s^m$, we claim the statement about the operators preserving $\mathcal{M}_s^m(A_v)$ is a consequence of (5.9)–(5.11). Indeed, in either case $\ell \neq \wp$ or $\ell = \wp$ we have $V_\ell(A_v[[u]]) \subseteq A_v[[u]]$, since in (5.10) the u_ℓ^n terms are in $A[[u]]$ by (4.4). Likewise for U_ℓ, the polynomials $G_{n,\Lambda_\ell}(\ell u)$ in (5.9) are in $A[u]$, as the $\mathbb{F}_q$-lattice Λ_ℓ has exponential function given by polynomials from the Carlitz action, namely $e_{\Lambda_\ell}(z) = C_\ell(z)/\ell$, and thus by Theorem 3.2(b),

$$\mathcal{G}_{\Lambda_\ell}(\ell u, x) = \sum_{n=1}^{\infty} G_{n,\Lambda_\ell}(\ell u) x^n = \frac{\ell u x}{1 - uC_\ell(x)} \in \ell \cdot A[u][[x]].$$

Additionally we recall that the cases of $U_\wp$ and $V_\wp$ preserving v-integrality were previously proved by Vincent [24, Cor. 3.2, Prop. 3.3].

Now by hypothesis we can choose a sequence $\{f_i\}$ of Drinfeld modular forms of weight k_i and type m so that $f_i \to f$ and $k_i \to s$. By Proposition 5.14(b), for any $i \geqslant 0$,

$$g_d^{q^i} \equiv 1 \pmod{\wp^{q^i}},$$

since $\mathrm{ord}_v([d]) = 1$. The form $g_d^{q^i}$ has weight $(q^d-1)q^i$ and type 0, and certainly $f_i g_d^{q^i} \to f$ with respect to the $\|\cdot\|_v$-norm. However, we also have that as real numbers,

$$\text{weight of } f_i g_d^{q^i} = k_i + (q^d-1)q^i \to \infty, \quad \text{as } i \to \infty.$$

Therefore, it suffices to assume that $k_i \to \infty$ as real numbers, as $i \to \infty$.

Suppose that $f = \sum c_n u^n$, $f_i = \sum c_{n,i} u^n \in K \otimes A_v[[u]]$. For $\ell \neq \wp$, since $\ell^{k_i} \to \ell^s$ and $c_{n,i} \to c_n$, we have

$$T_\ell(f_i) = \ell^{k_i} \sum_{n=0}^{\infty} c_{n,i} u_\ell^n + \sum_{n=0}^{\infty} c_{n,i} G_{n,\Lambda_\ell}(\ell u) \ \longrightarrow\ T_\ell(f).$$

Since $T_\ell(f_i) \in M_{k_i}^m(K)$, it follows that $T_\ell(f) \in \mathcal{M}_s^m$.

Now consider the case $\ell = \wp$. Since $k_i \to \infty$, we see that $|\wp^{k_i}|_v \to 0$. Therefore,

$$T_\wp(f_i) \to \sum_{n=0}^{\infty} c_n G_{n,\Lambda_\wp}(\wp u) = U_\wp(f),$$

and so $U_\wp(f) \in \mathcal{M}_s^m$. By the same argument each $U_\wp(f_i) \in \mathcal{M}_{k_i}^m$, starting with the constant sequence f_i in the first paragraph. By subtraction each

$$V_\wp(f_i) = \wp^{-k_i}\bigl(T_\wp(f_i) - U_\wp(f_i)\bigr) \tag{5.15}$$

is then an element of $\mathcal{M}_{k_i}^m$. Because $c_{n,i} \to c_n$, we see from (5.10) that $V_\wp(f_i) \to V_\wp(f)$ with respect to the $\|\cdot\|_v$-norm. Thus by Proposition 5.8, $V_\wp(f) \in \mathcal{M}_s^m$ as desired. □

6. Theta operators on v-adic modular forms

As is well known the operators Θ^r do not generally take Drinfeld modular forms to Drinfeld modular forms [1], [22]. However, we will prove in this section that each Θ^r, $r \geqslant 0$, does preserve spaces of v-adic modular forms. Using the equivalent formulations in (4.12a) and (4.12b), we define K_v-linear operators

$$\Theta^r : K \otimes A_v[[u]] \to K \otimes A_v[[u]], \quad r \geqslant 0.$$

Theorem 6.1. *For any weight $s \in \mathbb{S}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, we have for $r \geqslant 0$,*

$$\Theta^r : \mathcal{M}_s^m \to \mathcal{M}_{s+2r}^{m+r}.$$

This can be seen as similar in spirit to the results of Bosser and Pellarin [1, Thm. 2], [2, Thm. 4.1] (see also Theorem 4.14), that Θ^r preserves spaces of Drinfeld quasi-modular forms, and our main arguments rely on essentially showing that quasi-modular forms with K_v-coefficients are v-adic and applying Theorem 4.14. Consider first the operator $\Theta = \Theta^1$, which can be equated by (2.12) with the operation on u-expansions given by

$$\Theta = u^2\, \partial_u^1.$$

We recall a formula of Gekeler [7, §8] (take $r = 1$ in (4.13)), which states that for $f \in M_k^m$,

$$\Theta(f) = \mathcal{D}^1(f) + kEf,$$

where E is the false Eisenstein series whose u-expansion is given in (4.8). Our first goal is to show that E is a v-adic modular form, for which we use results of Goss and Petrov.

For k, $n \geqslant 1$ and $s \in \mathbb{S}$, we set

$$f_{k,n} := \sum_{a \in A_+} a^{k-n} G_n(u_a), \quad \hat{f}_{s,n} := \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a^s G_n(u_a). \tag{6.2}$$

The notation $f_{k,n}$ and $\hat{f}_{s,n}$ is not completely consistent, since $f_{k,n}$ is more closely related to $\hat{f}_{k-n,n}$ than $\hat{f}_{k,n}$, but this viewpoint is convenient in many contexts (see [14]).

Theorem 6.3 (Goss [14, Thm. 2], Petrov [19, Thm. 1.3])**.**

(a) (Petrov) *Let k, $n \geqslant 1$ be chosen so that $k - 2n > 0$, $k \equiv 2n \pmod{q-1}$, and $n \leqslant p^{\mathrm{ord}_p(k-n)}$. Then*
$$f_{k,n} \in M_k^m(K),$$
where $m \equiv n \pmod{q-1}$.

(b) (Goss) *Let $n \geqslant 1$. For $s = (x, y) \in \mathbb{S}$ with $x \equiv n \pmod{q-1}$ and $y \equiv 0 \pmod{q^{\lceil \log_q(n) \rceil}}$, we have*
$$\hat{f}_{s,n} \in \mathcal{M}_{s+n}^m,$$
where $m \equiv n \pmod{q-1}$.

We note that the statement of Theorem 6.3(b) is slightly stronger than what is stated in [14], but Goss' proof works here without changes. We then have the following corollary.

Corollary 6.4. *For any $\ell \equiv 0 \pmod{q-1}$, we have $\hat{f}_{\ell+1,1} \in \mathcal{M}_{\ell+2}^1$.*

If we take $\ell = 0$, we see that
$$\hat{f}_{1,1} = \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a \in A_v[[u]]$$
is a v-adic modular form in $\mathcal{M}_2^1(A_v)$ and is a partial sum of E in (4.8). From this we can prove that E itself is a v-adic modular form.

Theorem 6.5. *The false Eisenstein series E is a v-adic modular form in $\mathcal{M}_2^1(A_v)$.*

Proof. Starting with the expansion in (4.8), we see that $E \in A_v[[u]]$. Also,
$$\begin{aligned} E = \sum_{a \in A_+} a u_a &= \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a + \wp \sum_{a \in A_+} a u_{\wp a} \\ &= \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a + \wp \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_{\wp a} + \wp^2 \sum_{a \in A_+} a u_{\wp^2 a}, \end{aligned}$$
and continuing in this way, we find
$$E = \sum_{j=0}^{\infty} \left(\wp^j \sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_{\wp^j a} \right).$$
We note that
$$\sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_{\wp^j a} = V_\wp^{\circ j} \left(\sum_{\substack{a \in A_+ \\ \wp \nmid a}} a u_a \right) = V_\wp^{\circ j}(\hat{f}_{1,1}),$$

where $V_\wp^{\circ j}$ is the j-th iterate $V_\wp \circ \cdots \circ V_\wp$. By Proposition 5.12, we see that $V_\wp^{\circ j}(\hat{f}_{1,1}) \in \mathcal{M}_2^1(A_v)$ for all j. Moreover,

$$E = \sum_{j=0}^{\infty} \wp^j V_\wp^{\circ j}(\hat{f}_{1,1}),$$

the right-hand side of which converges with respect to the $\|\cdot\|_v$-norm, and so we are done by Proposition 5.8. □

Proof of Theorem 6.1. Let $f \in \mathcal{M}_s^m$ and pick $f_i \in M_{k_i}^m(K)$ with $f_i \to f$. It follows from the formulas in Corollary 4.12 that $\Theta^r(f_i) \to \Theta^r(f)$ with respect to the $\|\cdot\|_v$-norm for each $r \geqslant 0$, so by Proposition 5.8 it remains to show that each

$$\Theta^r(f_i) \in \mathcal{M}_{k_i+2r}^{m+r}.$$

We proceed by induction on r. If $r = 1$, then since $\mathcal{D}^1(f_i) \in M_{k_i+2}^{m+1}(K)$ for each i by Theorem 4.14, it follows from Theorem 6.5 that

$$\Theta(f_i) = \mathcal{D}^1(f_i) + k_i E f_i \in \mathcal{M}_{k_i+2}^{m+1},$$

for each i. Now by (4.13), for each i

$$\Theta^r(f_i) = \mathcal{D}^r(f_i) - \sum_{j=1}^{r} (-1)^j \binom{k_i + r - 1}{j} \Theta^{r-j}(f_i)\Theta^{j-1}(E).$$

By Theorem 4.14, $\mathcal{D}^r(f_i) \in M_{k_i+2r}^{m+r}(K)$, and by the induction hypothesis and Theorem 6.5 the terms in the sum are in $\mathcal{M}_{k_i+2r}^{m+r}$. □

7. Theta operators and v-adic integrality

We see from Theorem 6.1 that $\Theta^r : \mathcal{M}_s^m \to \mathcal{M}_{s+2r}^{m+r}$, and it is a natural question to ask whether Θ^r preserves v-integrality, i.e.,

$$\Theta^r : \mathcal{M}_s^m(A_v) \stackrel{?}{\to} \mathcal{M}_{s+2r}^{m+r}(A_v).$$

However, it is known that this can fail for r sufficiently large because of the denominators in $G_{r+1}(u)$ (e.g., see Vincent [25, Cor. 1]). Nevertheless, in this section we see that Θ^r is not far off from preserving v-integrality.

For an A-algebra R and a sequence $\{b_m\} \subseteq R$, we define an A-Hurwitz series over R (cf. [13, §9.1]) by

$$h(x) = \sum_{m=0}^{\infty} \frac{b_m}{\Pi_m} x^m \in (K \otimes_A R)[[x]], \tag{7.1}$$

where we recall the definition of the Carlitz factorial Π_m from (2.3). Series of this type were initially studied by Carlitz [4, §3] and further investigated by Goss [9, §3], [13, §9.1]. The particular cases we are interested in are when $R = A$ or $R = A[u]$, but we have the following general proposition whose proof can be easily adapted from [9, §3.2], [13, Prop. 9.1.5].

Proposition 7.2. *Let R be an A-algebra, and let $h(x)$ be an A-Hurwitz series over R.*

(a) *If the constant term of $h(x)$ is 1, then $1/h(x)$ is also an A-Hurwitz series over R.*
(b) *If $g(x)$ is an A-Hurwitz series over R with constant term 0, then $h(g(x))$ is also an A-Hurwitz series over R.*

We apply this proposition to the generating function of Goss polynomials.

Lemma 7.3. *For each $k \geqslant 1$, we have $\Pi_{k-1} G_k(u) \in A[u]$.*

Proof. Consider the generating series

$$\frac{\mathcal{G}(u,x)}{x} = \sum_{k=1}^{\infty} G_k(u) x^{k-1} = \frac{u}{1 - ue_C(x)}.$$

We claim that $\mathcal{G}(u,x)/x$ is an A-Hurwitz series over $A[u]$. Indeed certainly the constant series u itself is one, and

$$1 - ue_C(x) = 1 - \sum_{i=0}^{\infty} \frac{ux^{q^i}}{\Pi_{q^i}}$$

is an A-Hurwitz series over $A[u]$ with constant term 1, so the claim follows from Proposition 7.2(a). The result is then immediate. □

Theorem 7.4. *For $r \geqslant 0$, if $f \in \mathcal{M}_s^m(A_v)$, then $\Pi_r \Theta^r(f) \in \mathcal{M}_{s+2r}^{m+r}(A_v)$. Thus we have a well-defined operator,*

$$\Pi_r \Theta^r : \mathcal{M}_s^m(A_v) \to \mathcal{M}_{s+2r}^{m+r}(A_v).$$

Proof. By (4.12a), we see that the possible denominators of $\Theta^r(f)$ come from the denominators of $G_{r+1}(u)$, which are cleared by Π_r using Lemma 7.3. □

Remark 7.5. Once we see that $\Pi_r \Theta^r$ preserves v-integrality, the question of whether Π_r is the best possible denominator is important but subtle, and in general the answer is no. For example, taking $r = q^{d+1} - 1$, we see from Theorem 3.2(d), that

$$G_{q^{d+1}}(u) = u^{q^{d+1}},$$

and so $\Theta^{q^{d+1}-1} : A_v[[u]] \to A_v[[u]]$ already by (4.12a). However, $\Pi_{q^{d+1}-1}$ can be seen to be divisible by $\wp$.

Nevertheless, we do see that Π_r is the best possible denominator in many cases. For example, let $r = q^i$ for $i \geqslant 1$. Then from Theorem 3.2(a),

$$G_{q^i+1}(u) = u\left(G_{q^i}(u) + \frac{G_{q^i+1-q}(u)}{D_1} + \cdots + \frac{G_1(u)}{D_i} \right) = u\left(u^{q^i} + \frac{G_{q^i+1-q}(u)}{D_1} + \cdots + \frac{u}{D_i} \right).$$

From Theorem 3.2(b) we know that u^2 divides $G_k(u)$ for all $k \geqslant 2$, and so we find that the coefficient of u^2 in $G_{q^i+1}(u)$ is precisely $1/D_i$, which is the same as Π_{q^i}.

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