1. After trigonometric substitution, the integral
\[ \int \frac{\sqrt{x^2 - 4}}{x} \, dx \]
becomes
(A) \( 4 \int \frac{\tan^3 \theta}{\sec \theta} \, d\theta \)
(B) \( 2 \int \tan^2 \theta \, d\theta \)
(C) \( 8 \int \frac{\sec^3 \theta}{\tan \theta} \, d\theta \)
(D) \( 4 \int \frac{\cos \theta}{\sin \theta} \, d\theta \)
(E) \( 2 \int \cos^2 \theta \sin \theta \, d\theta \)

**Solution:** [B] Let \( x = 2 \sec \theta \) so that \( x^2 - 4 = 4 \tan^2 \theta \) and \( dx = 2 \sec \theta \tan \theta \, d\theta \). We substitute:

\[ \int \frac{\sqrt{x^2 - 4}}{x} \, dx = \int \frac{\sqrt{4 \tan^2 \theta}}{2 \sec \theta} \, 2 \sec \theta \tan \theta \, d\theta = 2 \int \tan^2 \theta \, d\theta. \]

2. Consider the following improper integrals:

I. \( \int_0^1 \frac{1}{x^2} \, dx \)  
II. \( \int_0^\infty \frac{1}{x^2} \, dx \)  
III. \( \int_1^\infty \frac{1}{x^2} \, dx \)

Which of these integrals converges?

(A) I, II, and III
(B) I and III only
(C) I only
(D) II only
(E) III only

**Solution:** [E] We compute

\[ \int_0^1 \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_0^1 = \lim_{a \to 0^+} \left[ -1 + \frac{1}{a} \right] = \infty, \]
so I diverges. Since the interval of integration for II contains [0, 1], II also diverges.

By process of elimination we rule out A–D, so the answer must be E. However, we also compute

\[ \int_1^\infty \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_1^\infty = \lim_{b \to \infty} \left[ -\frac{1}{b} + 1 \right] = 1, \]

and III converges.

3. The partial fraction decomposition of

\[ \frac{1}{(x^2 - 1)^2} \]

has the form

(A) \( \frac{A}{x-1} + \frac{B}{x+1} \)

(B) \( \frac{Ax + B}{x^2 - 1} + \frac{Cx + D}{(x^2 - 1)^2} \)

(C) \( \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \)

(D) \( \frac{A}{x-1} + \frac{Bx + C}{(x-1)^2} + \frac{D}{x+1} + \frac{E x + F}{(x+1)^2} \)

(E) \( \frac{Ax + B}{(x-1)^2} + \frac{Cx + D}{(x+1)^2} \)

**Solution:** \( \boxed{C} \) We factor the denominator as

\[
\frac{1}{(x^2 - 1)^2} = \frac{1}{((x-1)(x+1))^2} = \frac{1}{(x-1)^2(x+1)^2}.
\]

Thus the partial fraction decomposition is C.

4. If we use Simpson’s rule to approximate

\[ \int_0^2 \sin(x) \, dx \]

with 6 equal subintervals, we obtain which expression below?

(A) \( \frac{1}{9} \left[ \sin(0) + 4 \sin\left(\frac{1}{3}\right) + 2 \sin\left(\frac{2}{3}\right) + 4 \sin(1) + 2 \sin\left(\frac{4}{3}\right) + 4 \sin\left(\frac{5}{3}\right) + \sin(2) \right] \)

(B) \( \frac{1}{9} \left[ \sin(0) + 2 \sin\left(\frac{1}{3}\right) + 4 \sin\left(\frac{2}{3}\right) + 2 \sin(1) + 4 \sin\left(\frac{4}{3}\right) + 2 \sin\left(\frac{5}{3}\right) + \sin(2) \right] \)

(C) \( \frac{1}{6} \left[ \sin(0) + 2 \sin\left(\frac{1}{3}\right) + 2 \sin\left(\frac{2}{3}\right) + 2 \sin(1) + 2 \sin\left(\frac{4}{3}\right) + 2 \sin\left(\frac{5}{3}\right) + \sin(2) \right] \)

(D) \( \frac{1}{6} \left[ \sin(0) + 2 \sin\left(\frac{1}{3}\right) + 4 \sin\left(\frac{2}{3}\right) + 4 \sin(1) + 4 \sin\left(\frac{4}{3}\right) + 2 \sin\left(\frac{5}{3}\right) + \sin(2) \right] \)
(E) $\frac{1}{3} \left[ \sin(0) + \sin\left(\frac{1}{3}\right) + \sin\left(\frac{2}{3}\right) + \sin(1) + \sin\left(\frac{4}{3}\right) + \sin\left(\frac{5}{3}\right) + \sin(2) \right]$

**Solution:** [A] It is a matter of checking the formula for Simpson’s rule against the choices given.

5. Evaluate $\lim_{n \to \infty} \frac{n \cos((2n + 1)\pi)}{2n - 1}$.

   (A) $\frac{1}{2}$
   (B) 1
   (C) $-1$
   (D) $-\frac{1}{2}$
   (E) does not exist

**Solution:** [D] We note that for $n$ an integer, the expression $2n + 1$ is always an odd integer. Also,$\cos(\pi) = \cos(3\pi) = \cos(5\pi) = \cdots = \cos((2n + 1)\pi) = -1$,

and thus

$$\lim_{n \to \infty} \frac{n \cos((2n + 1)\pi)}{2n - 1} = \lim_{n \to \infty} \frac{-n}{2n - 1} = -\frac{1}{2}.$$

6. The series $\sum_{n=0}^{\infty} \frac{3^n - 2}{4^n}$

   (A) converges to 0
   (B) converges to 2
   (C) converges to $\frac{4}{3}$
   (D) converges to $\frac{8}{3}$
   (E) diverges

**Solution:** We rewrite the series

$$\sum_{n=0}^{\infty} \frac{3^n - 2}{4^n} = \sum_{n=0}^{\infty} \frac{3^n}{4^n} - \sum_{n=0}^{\infty} \frac{2}{4^n}.$$

Both of the series on the right are geometric series, and continuing we find

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n} - \sum_{n=0}^{\infty} \frac{2}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n - \sum_{n=0}^{\infty} 2 \left(\frac{1}{4}\right)^n.$$
Each series converges since $|r| = \frac{3}{4} < 1$ for the first and $|r| = \frac{1}{2} < 1$ for the second, and by the formula for a convergent geometric series,

$$
\sum_{n=0}^{\infty} \frac{3^n - 2}{4^n} = \frac{1}{1 - \frac{3}{4}} - \frac{2}{1 - \frac{1}{4}} = \frac{4}{1} - \frac{8}{3} = \frac{4}{3}
$$

7. Evaluate the following integrals.

(a) \[ \int \frac{3x + 9}{x^2 + x - 2} \, dx \]

(b) \[ \int \frac{1}{(4 - 9x^2)^{3/2}} \, dx \]

(c) \[ \int_{2}^{\infty} xe^{1-x^2} \, dx \]

**Solution:** (a) We use the method of partial fractions. The denominator factors as

$$x^2 + x - 2 = (x + 2)(x - 1),$$

so the partial fraction decomposition will be of the form

$$\frac{3x + 9}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1}.$$

To find $A$ and $B$ we recombine to find

$$A \left( \frac{1}{x + 2} \right) + B \left( \frac{1}{x - 1} \right) = \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)},$$

and so we must have

$$A(x - 1) + B(x + 2) = 3x + 9.$$

Substituting in $x = 1$, we see that $3B = 12$, so $B = 4$. Substituting in $x = -2$, we see that $-3A = 3$, so $A = -1$. Therefore,

$$\int \frac{3x + 9}{x^2 + x - 2} \, dx = \int \left( \frac{-1}{x + 2} + \frac{4}{x - 1} \right) \, dx = -\ln |x + 2| + 4 \ln |x - 1| + C.$$

(b) We use trigonometric substitution. Take $x = \frac{2}{3} \sin \theta$ so that $4 - 9x^2 = 4 - 4\sin^2 \theta = 4\cos^2 \theta$ and $dx = \frac{2}{3} \cos \theta \, d\theta$. Substituting we find

$$\int \frac{1}{(4 - 9x^2)^{3/2}} \, dx = \int \frac{1}{(4 \cos^2 \theta)^{3/2}} \cdot \frac{2}{3} \cos \theta \, d\theta = \frac{2}{3} \int \frac{\cos \theta}{8 \cos^3 \theta} \, d\theta = \frac{1}{12} \int \frac{1}{\cos^2 \theta} \, d\theta.$$

To integrate this we recall that $\frac{1}{\cos \theta} = \sec \theta$, so we now have

$$\int \frac{1}{(4 - 9x^2)^{3/2}} \, dx = \frac{1}{12} \int \sec^2 \theta \, d\theta = \frac{1}{12} \tan \theta + C.$$
Since $\sin \theta = \frac{3x}{2}$, we consider a right triangle with angle $\theta$, opposite side $3x$, and hypotenuse 2. The third side then has length $\sqrt{4 - 9x^2}$, and so

$$\tan \theta = \frac{3x}{\sqrt{4 - 9x^2}}.$$ 

From this we see that

$$\int \frac{1}{(4 - 9x^2)^{3/2}} \, dx = \frac{1}{12} \cdot \frac{3x}{\sqrt{4 - 9x^2}} + C = \frac{x}{4\sqrt{4 - 9x^2}} + C.$$

(c) We use a $u$-substitution. Taking $u = 1 - x^2$, $du = -2x \, dx$, we have

$$\int xe^{1-x^2} \, dx = -\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{1-x^2} + C.$$

Thus

$$\int_2^\infty xe^{1-x^2} \, dx = \left[ -\frac{1}{2} e^{1-x^2} \right]_2^\infty = \lim_{b \to \infty} \left[ -\frac{1}{2} e^{1-b^2} + \frac{1}{2} e^{-3} \right] = 0 + \frac{1}{2} e^{-3} = \frac{1}{2e^3}.$$ 

8. Write down an integral for the length of the curve defined by $y = \sqrt{x}$ between the points $(0,0)$ and $(4,2)$. \textbf{(You need not evaluate the integral.)}

\textbf{Solution:} Using the arclength formula, we have

$$L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^4 \sqrt{1 + \left( \frac{1}{2\sqrt{x}} \right)^2} \, dx = \int_0^4 \sqrt{1 + \frac{1}{4x}} \, dx.$$ 

9. Solve the following initial value problems.

(a) $\frac{dy}{dx} = x + xy^2$, \hspace{0.5cm} \text{y(0) = 1}$

(b) $y' = e^{2x} - 3y$, \hspace{0.5cm} \text{y(0) = 1}$

\textbf{Solution:} (a) We use separation of variables. We see that

$$\frac{dy}{dx} = x(1 + y^2),$$

so

$$\frac{dy}{1 + y^2} = x \, dx.$$

Integrating both sides,

$$\int \frac{dy}{1 + y^2} = \int x \, dx \quad \Rightarrow \quad \tan^{-1}(y) = \frac{x^2}{2} + C.$$ 

Thus taking tangent of both sides,

$$y(x) = \tan \left( \frac{x^2}{2} + C \right).$$
Now \( y(0) = 1 \) implies that
\[
1 = \tan(C) \iff C = \frac{\pi}{4},
\]
so
\[
y(x) = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right).
\]

(b) The equation is first order linear. We rewrite it as
\[
y' + 3y = e^{2x},
\]
and so \( P = 3 \) and \( Q = e^{2x} \). The integrating factor \( I(x) \) is
\[
I(x) = \int e^{3x} \, dx = e^{3x}.
\]
Therefore, from the formula for first order linear equations,
\[
y(x) = \frac{1}{I(x)} \int I(x)Q(x) \, dx = \frac{1}{e^{3x}} \int e^{3x} \cdot e^{2x} \, dx = e^{-3x} \int e^{5x} \, dx = e^{-3x} \left( \frac{1}{5} e^{5x} + C \right).
\]
Since \( y(0) = 1 \), we have
\[
1 = \frac{1}{5} + C \iff C = \frac{4}{5},
\]
and so
\[
y(x) = \frac{1}{5} e^{2x} + \frac{4}{5} e^{-3x}.
\]

10. For each of the following sequences, find its limit or show that it does not exist. Justify your work.

(a) \( a_n = \frac{e^{2n}}{4^n} \)

(b) \( b_n = \frac{\cos(n\pi)}{n^2} \)

(c) \( c_n = \left(1 - \frac{2}{n}\right)^{n/2} \)

**Solution:**

(a) We observe that
\[
\lim_{n \to \infty} \frac{e^{2n}}{4^n} = \lim_{n \to \infty} \left( \frac{e^2}{4} \right)^n.
\]
Since \( e^2 > 4 \), the limit on the right will be \( \infty \) and so the limit does not exist.

(b) For different integers \( n \), we know that \( \cos(n\pi) = \pm 1 \), so the denominator should overwhelm the numerator giving a value of 0. To see this, we observe that
\[
-1 \leq \cos(n\pi) \leq 1 \iff -\frac{1}{n^2} \leq \frac{\cos(n\pi)}{n^2} \leq \frac{1}{n^2}.
\]
Both the left- and right-hand sides go to 0, so by the squeeze theorem
\[
\lim_{n \to \infty} \frac{\cos(n\pi)}{n^2} = 0.
\]

(c) We recall that for any real number \(c\),
\[
\lim_{n \to \infty} \left(1 + \frac{c}{n}\right)^n = e^c.
\]
Thus
\[
\lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^{n/2} = (e^{-2})^{1/2} = \frac{1}{\sqrt{e}}.
\]
Or we calculate:
\[
\lim_{n \to \infty} \ln \left(1 - \frac{2}{n}\right)^{n/2} = \lim_{n \to \infty} \frac{n}{2} \cdot \ln \left(1 - \frac{2}{n}\right)
\]
\[
= \lim_{n \to \infty} \frac{\ln \left(1 - \frac{2}{n}\right)}{\frac{2}{n}}
\]
\[
= \lim_{n \to \infty} \frac{\frac{1}{1 - \frac{2}{n} \cdot \frac{2}{n^2}}}{-\frac{2}{n^2}}
\]
\[
= -1.
\]
(In the third equality we have applied l’Hôpital’s rule.) Thus,
\[
\lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^{n/2} = \lim_{n \to \infty} e^{\ln \left(1 - \frac{2}{n}\right)^{n/2}} = e^{-1}.
\]