1. Find the formula for the \( n \)-th term of the sequence \( \{a_n\} \) that starts
\[
\begin{align*}
a_0 &= -\frac{1}{3}, & a_1 &= \frac{1}{4}, & a_2 &= -\frac{3}{7}, & a_3 &= \frac{9}{10}, & a_4 &= -\frac{27}{13}, & a_5 &= \frac{81}{16}.
\end{align*}
\]
(A) \( \frac{(-1)^n \cdot (3^n - 3^{n-1})}{3n - 1} \)
(B) \( \frac{(-1)^n \cdot 3^n}{n + 3} \)
(C) \( \frac{(-1)^{n+1} \cdot 3^{n-1}}{3n + 1} \)
(D) \( \frac{(-1)^{n+1} \cdot 3^{n+2}}{3n + 1} \)
(E) \( \frac{(-1)^{n+1} \cdot 3^{n+1}}{3n + 9} \)

**Solution:** C It is a simple matter to check that this is correct.

2. Evaluate \( \lim_{n \to \infty} 3 \cos(2\pi n) \left( \frac{1}{n} - 2 \right) \).
(A) 0
(B) 3
(C) \(-\frac{1}{2}\)
(D) \(-\frac{3}{2}\)
(E) The limit does not exist.

**Solution:** D The key thing is that, when \( n \) is an integer, \( 2n + 1 \) is an odd integer, and so \( \cos(2n\pi) = 1 \). So the limit is then \( \frac{3}{-2} = -\frac{3}{2} \).

3. The series \( \sum_{n=0}^{\infty} \frac{3^{2n} + (-2)^{n+1}}{10^n} \)
(A) converges to \( \frac{25}{3} \)
(B) converges to \( \frac{17}{2} \)
(C) converges to 8
(D) converges to 9
(E) diverges.

**Solution:** A This series can be decomposed into two geometric series:
\[
\sum_{n=0}^{\infty} \frac{3^{2n} + (-2)^{n+1}}{10^n} = \sum_{n=0}^{\infty} \left( \frac{3^2}{10} \right)^n + \sum_{n=0}^{\infty} -2 \cdot \left( \frac{-2}{10} \right)^n .
\]
Since both geometric series converge ($|\frac{9}{10}| < 1$ and $|\frac{-2}{10}| < 1$), from the formula for the sum of a geometric series,

$$
\sum_{n=0}^{\infty} \frac{3^{2n} + (-2)^{n+1}}{10^n} = \frac{1}{1 - \frac{9}{10}} - \frac{2}{1 + \frac{1}{5}} = \frac{25}{3}
$$

4. Consider the series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{11^n}$. By the Alternating Series Estimation Theorem, what is the smallest number of terms $m$ of this series that we would need to take so that the partial sum

$$
\sum_{n=1}^{m} (-1)^{n+1} \frac{1}{11^n}
$$

is within 0.001 of the value of the sum $S$?

(A) 1
(B) 2
(C) 3
(D) 4
(E) The series diverges.

**Solution:** B By the Alternating Series Estimation Theorem, the sum of the first $m$ terms is within $\frac{1}{11^{m+1}}$ of $S$. We therefore want the smallest $m$ for which

$$
\frac{1}{11^{m+1}} < 0.001 = \frac{1}{10^3}.
$$

So we need $m = 2$.

5. Which of the following integrals represents the surface area of the surface obtained by rotating the graph of $y = 2x^2$ over the interval $[1, 3]$ about the $x$-axis?

(A) $2\pi \int_{1}^{3} x \sqrt{1 + 16x^2} \, dx$
(B) $4\pi \int_{1}^{3} x^2 \sqrt{1 + 16x^2} \, dx$
(C) $4\pi \int_{1}^{3} x^2 \sqrt{1 - 16x^2} \, dx$
(D) $2\pi \int_{1}^{3} x \sqrt{1 - 4x} \, dx$
(E) $4\pi \int_{1}^{3} x^2 \sqrt{1 + 4x^4} \, dx$

**Solution:** B Using the formula for the surface area of a surface of revolution,

$$
S.A. = 2\pi \int_{1}^{3} 2x^2 \sqrt{1 + (4x)^2} \, dx,
$$

which is B.
6. Find the arclength of the graph of the curve defined by \( y = 2x^{3/2} \) from \( x = 0 \) to \( x = 5/3 \).

**Solution:** We use the arclength formula:

\[
\begin{align*}
s &= \int_0^{5/3} \sqrt{1 + (y')^2} \, dx \\
&= \int_0^{5/3} \sqrt{1 + (3\sqrt{x})^2} \, dx \\
&= \int_1^{16} \sqrt{u} \, du \quad (u = 1 + 9x, \ du = 9 \, dx) \\
&= \left[ \frac{2}{27}u^{3/2} \right]_1^{16} \\
&= \frac{2}{27}(16)^{3/2} - \frac{2}{27} = \frac{14}{3}.
\end{align*}
\]

7. Find the limits of the following sequences.

(a) \( a_n = \frac{2n^3 - 3}{\sqrt{9n^6 + 20n}} \)

(b) \( b_n = \left( n - \frac{1}{n} \right)^n \)

**Solution:** (a) By comparing highest order terms in the numerator and denominator (both are degree 3), we see that the limit will be \( \frac{2}{\sqrt{9}} = \frac{2}{3} \).

(b) There are several ways to do this problem, but one is the following:

\[
\lim_{n \to \infty} \left( n - \frac{1}{n} \right)^n = \lim_{n \to \infty} \exp \left( n \ln \left( 1 - \frac{1}{n} \right) \right) \\
= \lim_{n \to \infty} \exp \left( \ln \left( \frac{1}{n} \right) \right) \\
= \exp \left( \lim_{n \to \infty} \frac{1}{1 - \frac{1}{n}} \left( \frac{1}{n^2} \right) \right) \\
= \exp (-1) = \frac{1}{e}.
\]

8. [10 points] Consider the series

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2n+1} - \frac{1}{2n+3} \right).
\]

(a) Find a formula for the partial sum \( S_m \) of the series.

(b) Find the sum of the series or show that it diverges.
Solution: (a) This is a telescoping series. After canceling the intermediate terms we have

$$ S_m = \frac{1}{3} - \frac{1}{2m + 3}. $$

(b) To find the sum of the series we take the limit of $S_m$ as $m \to \infty$:

$$ \lim_{m \to \infty} S_m = \lim_{m \to \infty} \left( \frac{1}{3} - \frac{1}{2m + 3} \right) = \frac{1}{3}. $$

9. Determine whether each series below converges or diverges. For each problem use one or more of the tests we have studied: absolute convergence test, alternating series test, direct comparison test, geometric series test, integral test, limit comparison test, $p$-series test, ratio test, or test for divergence ($n$-th term test). Be sure to specify which test(s) you are using and show how they apply.

(a) $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{1/n}$

(c) $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$

Solution: (a) First we observe that $|\cos(n)/n^2| \leq \frac{1}{n^2}$ for all $n$, and since the series $\sum \frac{1}{n^2}$ converges (it is a $p$-series with $p = 2 > 1$), by the **direct comparison test**, the series $\sum |\cos(n)/n^2|$ converges. Therefore, by the **absolute convergence test**, the series $\sum \frac{\cos(n)}{n^2}$ converges.

(b) Because

$$ \lim_{n \to \infty} \left(\frac{1}{2}\right)^{1/n} = \left(\frac{1}{2}\right)^0 = 1 \neq 0, $$

the **test for divergence** ($n$-th term test) implies that the series must diverge.

(c) We apply the **ratio test**:

$$ \rho = \lim_{n \to \infty} \frac{(3(n+1))!}{((n+1)!)^3} \cdot \frac{(3n)!}{(3n+3)(3n+2)(3n+1)} \cdot \frac{(3n)!}{(3n+3)(3n+2)(3n+1)} = 27 > 1, $$

and so the ratio test implies that the series diverges.

10. Consider the following function:

$$ f(x) = \sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n} \cdot 3^n}. $$

(a) What is the open interval of convergence for $f(x)$?

(b) Does the series converge at the endpoints of the interval in (a)? Why or why not?
(c) Write down a power series expression for \( g(x) = f''(x) \).

**Solution:** (a) We apply the ratio test:

\[
\lim_{n \to \infty} \frac{|x + 1|^{n+1}}{\sqrt{n+1} \cdot 3^{n+1}} \cdot \frac{3^n \sqrt{n}}{|x + 1|^n} = \lim_{n \to \infty} \frac{|x + 1|}{3} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x + 1|}{3}.
\]

So the open interval of convergence is defined by

\[
\frac{|x + 1|}{3} < 1 \iff -3 < x + 1 < 3 \iff -4 < x < 2.
\]

(b) We check at the endpoints. When \( x = 2 \), the series we obtain is

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},
\]

which diverges as it is a p-series with \( p = 1/2 \leq 1 \). When \( x = -4 \), the series we obtain is

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},
\]

which converges by the alternating series test.

(c) We differentiate the series for \( f(x) \) term-by-term and find

\[
f''(x) = \sum_{n=2}^{\infty} \frac{n(n-1)(x+1)^{n-2}}{\sqrt{n} \cdot 3^n}.
\]