

Exam #1 Solutions

1. (a) open sentence; possible true statements vary, but one example: "If $a = 3$, $b = 4$, and $c = 5$, then $a^2 + b^2 = c^2$."
 (b) neither
 (c) statement
 (d) open sentence (radius and circumference of what? a circle? a sphere?); Again answers may vary, try "Given a circle, the ratio of the circumference to the diameter is π ."
 (e) open sentence (we don't know what a , b , and c are); "If a , b , and c are real numbers and $a \neq 0$, then for every real number x , if $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$."
2. (a) For all $y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ so that $(x + y)^2 \neq x^2 + y^2$.
 (b) "if $y > 0$ then $y = 2^x$ " is equivalent to " $y \leq 0$ or $y = 2^x$." So the negation we're looking for is "There is a $y \in \mathbb{R}$ so that for all $x \in \mathbb{R}$, $y > 0$ and $y \neq 2^x$."
3. (a) In words this statement says roughly that every $a \in A$ is divisible by some $b \in B$, which is True.
 (b) This says that there is a $b \in B$ that divides every $a \in A$: False (3 does not divide 10, and 5 does not divide 6).
 (c) This says every $b \in B$ divides some $a \in A$: True.
 (d) This says there is an $a \in A$ that is divisible by each $b \in B$: True (take $a = 15$).
4. (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}$, if $x < y$, then $\exists z \in \mathbb{R}$ so that $x < z < y$.
 (b) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}$, if $x < 0$ and $y < 0$, then $xy > 0$.
 (c) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ so that $m > n$.
5. The blanks are as follows: $\in \mathbb{Z}$ be arbitrary; $n = 2k$; $4k^2$; k^2 ; 0; n is odd; $n = 2k + 1$; $(2k + 1)^2 = 4k^2 + 4k + 1$; $k^2 + k + \frac{1}{4}$; 1.
6. There are several ways to do this problem, either by manipulating the logical symbols algebraically or by writing out the truth tables. Or by a combination of both.
 (A) The truth table for $P \wedge (P \Rightarrow Q)$:

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Since $P \wedge (P \Rightarrow Q)$ has the same truth values as $P \wedge Q$, they are equivalent. So the answer is (J).

(B) We simplify a little bit: $P \neq Q \Rightarrow \neg P$ is equivalent to $Q \vee \neg P$, and $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$ by de Morgan's laws. Truth table:

P	Q	$Q \vee \neg P$	$\neg P \vee \neg Q$	$(\neg Q \Rightarrow \neg P) \wedge (\neg(P \wedge Q))$
T	T	T	F	F
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

This matches up with $\neg P$, so the answer is (I).

(C) We convert the implications to OR statements: so $((P \Rightarrow Q) \Rightarrow Q) \Rightarrow P$ is equivalent to

$$\begin{aligned} & \neg((P \Rightarrow Q) \Rightarrow Q) \vee P \\ \Leftrightarrow & \neg(\neg(P \Rightarrow Q) \vee Q) \vee P \\ \Leftrightarrow & \neg(\neg(\neg P \vee Q) \vee Q) \vee P, \end{aligned}$$

which we can simplify using de Morgan's laws to

$$((\neg P \vee Q) \wedge \neg Q) \vee P.$$

From here we can write down the truth table, but we can also do the following. We can simplify further to

$$((\neg P \vee Q) \wedge \neg Q) \vee P \Leftrightarrow ((\neg P \vee Q) \vee P) \wedge (\neg Q \vee P).$$

Since $\neg P \vee Q \vee P$ is *always true*, we see that

$$((P \Rightarrow Q) \Rightarrow Q) \Rightarrow P \Leftrightarrow \neg Q \vee P,$$

which is equivalent to $Q \Rightarrow P$. So the answer is (G).