Math 220

Exam #2 Solutions

1. Negation ($P \wedge \neg Q$): $\exists u \in \mathbb{Z}$ so that ($\exists m \in \mathbb{Z}$ so that $u = 4m+3$) and ($\exists a, b \in \mathbb{Z}$ so that $u = a^2 + b^2$).

   Converse ($Q \Rightarrow P$): $\forall u \in \mathbb{Z}$, if there are no $a, b \in \mathbb{Z}$ so that $u = a^2 + b^2$, then there is an $m \in \mathbb{Z}$ so that $u = 4m+3$.

   Contrapositive ($\neg Q \Rightarrow \neg P$): $\forall u \in \mathbb{Z}$, if $\exists a, b \in \mathbb{Z}$ so that $u = a^2 + b^2$, then $\forall m \in \mathbb{Z}$, $u \neq 4m+3$.

2. The key thing to do is see that $\neg(P \Leftrightarrow Q)$ is equivalent to $(P \wedge \neg Q) \vee (Q \wedge \neg P)$. Then the answer is: $\exists x \in \mathbb{R}$ so that $[(\forall y \in \mathbb{R}, \sin(y) \neq x)$ and $-1 < x < 1]$ or $[(\exists y \in \mathbb{R}$ so that $\sin(y) = x)$ and $(x \leq -1$ or $x \geq 1)]$.

3. The Well-Ordering Principle states: Every non-empty subset of $\mathbb{Z}^+$ contains a smallest element. In symbols this is, "For all non-empty subsets S of $\mathbb{Z}^+$, $\exists a \in S$ so that $\forall x \in S$, $a \leq x$." The negation of this is "There is a non-empty subset S of $\mathbb{Z}^+$ so that $\forall a \in S$, there is an $x \in S$ so that $a > x$." Other answers were also acceptable.

4. Let $a, b \in \mathbb{Z}$ be arbitrary. Then

$$\begin{aligned}
(a-b)+(b-a) &= (a+(-b))+(b+(-a)) && \text{(Definition of "$-$")} \\
&= (a+(-a))+(b+(-b)) && \text{(A1,A2)} \\
&= 0+0 && \text{(A4)} \\
&= 0 && \text{(A3).}
\end{aligned}$$

   Therefore, we have $(a-b)+(b-a) = 0$. Now add $-(a-b)$ to both sides:

$$-(a-b)+(a-b)+(b-a) = -(a-b)+0.$$

   Using A1, A3, and A4, we see that $b-a = -(a-b)$.

5. The contrapositive of this statement is $\forall x \in \mathbb{Z}$, if ($x < 0$ or $x > 10$), then $x^2 > 10x$. We'll prove this: Let $x \in \mathbb{Z}$ be arbitrary. Suppose that $x < 0$ or $x > 10$. This brings two cases:

   Case 1: $x < 0$. Then $x^2 > 0$ by Q5, and $10x < 0$ by Q4. By Q6, $x^2 > 10x$.

   Case 2: $x > 10$. Then $x > 0$ since $10 > 0$. Multiply by $x > 10$ by x (Q8) and obtain $x^2 > 10x$.

   So in both cases $x^2 > 10x$, which is what we wanted to show.

6. The completed statement is: For all $x \in \mathbb{Z}$, $x^3 - 4x^2 = 0$ if and only if $x = 0$ or $x = 4$.

   Proof. Let $x \in \mathbb{Z}$ be arbitrary.

   ($\Rightarrow$): Suppose that $x^3 - 4x^2 = 0$. By the distribution law, $x^2(x-4) = 0$. Therefore, by R2, we see that $x^2 = 0$ or $x = 4$. Again by R2, this means that $x = 0$ or $x = 4$, which is what we wanted to show.

   ($\Leftarrow$): Suppose that $x = 0$ or $x = 4$.

   Case 1: $x = 0$. Then $x^3 - 4x^2 = 0^3 - 4 \cdot 0^2 = 0 + 0 + 0$.

   Case 2: $x = 4$. Then $x^3 - 4x^2 = 4^3 - 4 \cdot 4^2 = 4^3 - 4^3 = 0$.

   So in both cases $x^3 - 4x^2 = 0$, which is what we wanted to show.

7. By the Binomial Theorem, $(8y-5x)^{203} = \displaystyle\sum_{k=0}^{203} \binom{203}{k} (8y)^{203-k}(-5x)^k$. Taking $k = 100$, we see that the coefficient of $x^{100}y^{103}$ is $\binom{203}{100} \cdot 8^{103} \cdot (-5)^{100}$.

8. For $n \in \mathbb{Z}^+$, let $P(n)$ denote the sentence "$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$." We will prove $P(n)$ is true by induction.

   Base case: $n = 1$. In this case the left-hand side of $P(1)$ is $\frac{1}{1\cdot 2} = \frac{1}{2}$. The right-hand side of $P(1)$ is $\frac{1}{1+1} = \frac{1}{2}$. Both sides are $\frac{1}{2}$, so $P(1)$ is true.

   Induction step: Let $k \in \mathbb{Z}^+$, and suppose that $P(k)$ is true. That is, we take as given that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$. We want to prove $P(k+1)$ is true. The left-hand side of $P(k+1)$ is

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)},$$

   and the sum $\frac{1}{1\cdot 2} + \cdots + \frac{1}{k(k+1)}$ is just the left-hand side of $P(k)$. Therefore, the left-and side of $P(k+1)$ is

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}.
\end{aligned}$$

   But this last expression is the right-hand side of $P(k+1)$. Thus, $P(k+1)$ is true (its left-hand side equals its right-hand side).

   By induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

9. For $n \in \mathbb{Z}^+$, let $P(n)$ denote the sentence "$n^3 - n$ is divisible by 3." Proceed by induction on n:

Base case: $n = 1$. $P(1)$ is the sentence "$1^3 - 1$ is divisible by 3." Of course, $1^3 - 1 = 0$ which is divisible by 3 (since $\frac{0}{3} = 0 \in \mathbb{Z}$). Therefore, $P(1)$ is true.

Induction step: Let $k \in \mathbb{Z}^+$ and suppose that $P(k)$ is true. Therefore $k^3 - k$ is divisible by 3: that is, $\frac{k^3-k}{3} \in \mathbb{Z}$. Now we want to show that $P(k+1)$ is true: that is, that $(k+1)^3 - (k+1)$ is divisible by 3. Now

$$\begin{aligned} \frac{(k+1)^3 - (k+1)}{3} &= \frac{k^3 + 3k^2 + 3k + 1 - k - 1}{3} \\ &= \frac{k^3 - k + 3k^2 + 3k}{3} \\ &= \frac{k^3 - k}{3} + k^2 + k. \end{aligned}$$

All three of these terms are integers (the first one since $P(k)$ is true). So their sum is an integer. Therefore $\frac{(k+1)^3-(k+1)}{3} \in \mathbb{Z}$, which is what we wanted to prove.

By induction, $n^3 - n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.