

Math 220
Exam 1 Sample Problems
September 21, 2003

The problems below should give you a sampling of what types of things to expect on the exam. One thing though, this is not meant to be an exhaustive list, and there may be types of problems on the exam that differ from these. Also, the exam is not likely to be as long as this list of problems. Enjoy!

1. Are the following sentences **statements**, **open sentences**, or **neither**? If a sentence is open, can you fix it so that it is a true statement?
 - (a) Every dog has its day.
 - (b) If $a > b$ and $b > c$, then $a > c$.
 - (c) $\frac{a^2 - b^2}{a - b} = a + b$.
 - (d) Let T be the trapezoid whose vertices are the points A , B , C , and D .
 - (e) Every integer that is a multiple of 6 is also a multiple of 4.
2. Negate the following statements:
 - (a) For all integers x , either x is odd or $\frac{x+1}{2}$ is not an integer.
 - (b) (Let E denote the set of even integers.) For all $a \in E$ and for all $b \in E$, 4 divides ab and 4 divides $a + b$.
 - (c) For all $x \in \mathbb{R}$, there is a $y \in \mathbb{R}$ so that $y^4 = x$.
 - (d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ so that $(y \neq 0) \Rightarrow (xy < 0)$.
 - (e) For all $n \in \mathbb{Z}^+$, if n is a prime number, then $2^n - 1$ is prime number.
3. For each of the original statements in (a)–(e) of problem 2, is the statement true or false? If a statement is false, explain why it is false.
4. In each of (a) and (b) below, there are two statements. In each case, one of the statements is true, and the other is false. Determine which statements are true and which are false, and explain why the false statements are indeed false.
 - (a) Let $[-1, 1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$.

“ $\forall x \in [-1, 1], \exists y \in \mathbb{R}$, so that $\sin(y) = x$.”
“ $\exists x \in [-1, 1]$ so that $\forall y \in \mathbb{R}, \sin(y) = x$.”
 - (b)

“ $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ so that $\frac{y^2-1}{y+1} = x$.”
“ $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+$ so that $\frac{y^2-1}{y+1} = x$.”

5. Negate the following statement forms:

(a) $(P \wedge \neg Q) \wedge (\neg P \vee Q)$

(b) $(\neg P \wedge Q) \vee (Q \Rightarrow P)$.

6. Construct the truth tables of the following statement forms. Are any of them tautologies or contradictions?

(a) $(P \vee Q) \wedge (\neg Q \vee \neg R)$

(b) $(P \wedge \neg Q) \vee (P \Rightarrow Q)$

(c) $[(P \wedge Q) \wedge (\neg P \vee R) \wedge (R \Rightarrow \neg Q)] \Rightarrow R$

7. Simplify the following statement forms into logically equivalent forms.

(a) $\neg((P \wedge Q) \vee (P \wedge \neg R))$

(b) $Q \wedge (P \vee \neg Q)$

(c) $\neg(\exists x \text{ so that } (P(x) \wedge \neg Q(x)) \vee (P(x) \wedge R(x)))$

(d) $\neg(\forall x, \exists y \text{ so that } P(x, y) \vee \neg R(x, y)) \vee (\exists x \text{ so that } \forall y, \neg P(x, y) \wedge Q(x, y))$

(e) $(P \Rightarrow Q) \wedge (R \vee Q)$

(f) $[(P \wedge Q) \wedge (\neg P \vee R) \wedge (R \Rightarrow \neg Q)] \Rightarrow R$

8. For all integers a and b , let “ $a \mid b$ ” be an abbreviation for “ a divides b .”

Fill in the blanks below in the proof of the statement,

$$“\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, \forall c \in \mathbb{Z}, \text{ if } a \mid b \text{ and } b \mid c, \text{ then } a \mid c.”$$

Proof. Let _____, _____, and _____ be arbitrary. Suppose that $a \mid b$ and $b \mid c$. (We now need to show that _____.) Because $a \mid b$, we conclude that $\frac{b}{a}$ is _____. Because $b \mid c$, we conclude that _____. Therefore,

$$\frac{c}{a} = \frac{b}{a} * \text{_____} .$$

Since $\frac{c}{a}$ is the product of two integers, we conclude that $\frac{c}{a}$ is also an _____. Therefore, _____, which is what we wanted to prove.

9. Here is a (true) fact: $\forall x \in \mathbb{R}^+, \exists n \in \mathbb{Z}^+$ so that $x < n$.

Fill in the blanks below in the proof of the statement,

$$\text{“}\forall x \in \mathbb{R}^+, \exists n \in \mathbb{Z}^+ \text{ so that } \frac{1}{n} < x < n.\text{”}$$

Proof. Let _____ be arbitrary. From the fact above, we can choose $n_1 \in \mathbb{Z}^+$ so that _____. Since $x > 0$, we also know that $\frac{1}{x} > 0$. Therefore from the fact above, we can choose $n_2 \in \mathbb{Z}^+$ so that _____ $> \frac{1}{x}$. Let n be the greater of _____ and _____. Then $n \geq n_1 > x$. Also,

$$n \geq n_2 > \text{_____} ,$$

and so inverting this inequality, we find

$$\frac{1}{n} \leq \text{_____} < \text{_____} .$$

Thus, we conclude _____, which is what we wanted to prove.