

Math 220
Exam 1 Sample Problems
Solutions Guide
September 22, 2003

1. (a) statement
(b) open sentence; you could write “For all real numbers a , b , and c , if $a > b$ and $b > c$, then $a > c$.” Answers may vary: could use integers instead of real numbers, etc.
(c) open sentence; you could write “For all real numbers a and b , if $a \neq b$, then $\frac{a^2-b^2}{a-b} = a + b$.” It is important to point out that $a \neq b$ in order for this to be a true statement. Another correct answer would be one of the form “For all real numbers a and b , if $a = 5$ and $b = 3$, then $\frac{a^2-b^2}{a-b} = a + b$.”
(d) neither
(e) statement
2. (a) There exists an integer x so that x is even and $\frac{x+1}{2}$ is an integer. Short version: $\exists x \in \mathbb{Z}$ so that x is even and $\frac{x+1}{2} \in \mathbb{Z}$.
(b) There exists $a \in E$ and there exists $b \in E$ so that either 4 does not divide ab or 4 does not divide $a + b$.
(c) There exists $x \in \mathbb{R}$ so that for all $y \in \mathbb{R}$, $y^4 \neq x$.
(d) $\exists x \in \mathbb{R}$ so that $\forall y \in \mathbb{R}$, both $y \neq 0$ and $xy \geq 0$. (Here we have used the fact that $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$, so “ $(y \neq 0) \Rightarrow (xy < 0)$ ” is logically equivalent to “ $y = 0$ and $xy < 0$.”)
(e) $\exists n \in \mathbb{Z}^+$ so that n is a prime number and $2^n - 1$ is not a prime number.
3. (a) True
(b) False: Let $a = 2$ and let $b = 4$. Then $ab = 8$, which is divisible by 4, but $a + b = 6$, which is not divisible by 4. (Both statements making up an “and” statement forms must be true for the whole thing to be true.)
(c) False: Let $x = -1$. Since for every $y \in \mathbb{R}$ we have $y^4 \geq 0$, you’ll never get $y^4 = -1$ for any $y \in \mathbb{R}$.
(d) True
(e) False: Let $n = 11$. Now 11 is prime, while $2^{11} - 1 = 2047 = 23 \cdot 89$ is not prime. (Note: you of course not be required on the exam to perform a calculation like this one.)
4. (a) The false statement is “ $\exists x \in [-1, 1]$ so that $\forall y \in \mathbb{R}$, $\sin(y) = x$.” In order for such an x to exist, the function $f(y) = \sin(y)$ would have to take on the value x for all y . No can do; the sine function is not constant.

- (b) The false statement is “ $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ so that $\frac{y^2-1}{y+1} = x$.” Indeed, let $x = -2$. As long as $y \neq -1$, we have

$$\frac{y^2-1}{y+1} = y-1, \quad y \neq -1.$$

If $y = -1$, then $\frac{y^2-1}{y+1}$ is undefined. Thus to find a $y \in \mathbb{R}$ so that $\frac{y^2-1}{y+1} = -2$, we can only look for $y \neq -1$, in which case $\frac{y^2-1}{y+1} = y-1 = -2$ if $y = -1$. So the only value of y that could possibly work is $y = -1$, but unfortunately, $\frac{y^2-1}{y+1}$ is not defined when $y = -1$. Thus no value of y will work.

5. (a) $(\neg P \vee Q) \vee (P \wedge \neg Q)$
 (b) $(P \vee \neg Q) \wedge (Q \wedge \neg P)$

6. (a)

P	Q	R	$P \vee Q$	$\neg Q \vee \neg R$	$(P \vee Q) \wedge (\neg Q \vee \neg R)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	T	F
F	F	F	F	T	F

It is not necessary to include the fourth and fifth columns, but it helps in understanding the last column.

- (b) Here it might be helpful, though not necessary, to use the fact that $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$.

P	Q	$P \wedge \neg Q$	$P \Rightarrow Q$ ($= \neg P \vee Q$)	$(P \wedge \neg Q) \vee (P \Rightarrow Q)$
T	T	F	T	T
T	F	T	F	T
F	T	F	T	T
F	F	F	T	T

Since the last column is all True, we see that $(P \wedge \neg Q) \vee (\neg P \vee Q)$ is a tautology.

(c) First of all, we have

P	Q	R	$P \wedge Q$	$\neg P \vee R$	$R \Rightarrow \neg Q$ ($= \neg R \vee \neg Q$)	$(P \wedge Q) \wedge (\neg P \vee R) \wedge (R \Rightarrow \neg Q)$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	T	F
T	F	F	F	F	T	F
F	T	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	T	T	F
F	F	F	F	T	T	F

It's worth pointing out in passing (though not required by the directions of the problem) that the right-most statement form is a contradiction. In any case, we find

P	Q	R	$(P \wedge Q) \wedge (\neg P \vee R) \wedge (R \Rightarrow \neg Q) \Rightarrow R$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

Thus the given statement form is a tautology.

7. Throughout these problems we use several times the following logical equivalences:

$$\begin{aligned} \neg(P \wedge Q) &\Leftrightarrow \neg P \vee \neg Q, \\ \neg(P \vee Q) &\Leftrightarrow \neg P \wedge \neg Q, \\ P \wedge (Q \vee R) &\Leftrightarrow (P \wedge Q) \vee (P \wedge R), \\ P \vee (Q \wedge R) &\Leftrightarrow (P \vee Q) \wedge (P \vee R), \\ P \Rightarrow Q &\Leftrightarrow \neg P \vee Q. \end{aligned}$$

(a)

$$\begin{aligned} \neg((P \wedge Q) \vee (P \wedge \neg R)) &\Leftrightarrow (\neg(P \wedge Q) \wedge \neg(P \wedge \neg R)) \\ &\Leftrightarrow (\neg P \vee \neg Q) \wedge (\neg P \vee R) \\ &\Leftrightarrow \boxed{\neg P \vee (\neg Q \wedge R)}. \end{aligned}$$

(b)

$$\begin{aligned} Q \wedge (P \vee \neg Q) &\Leftrightarrow (Q \wedge P) \vee (Q \wedge \neg Q) \\ &\Leftrightarrow \boxed{Q \wedge P}. \end{aligned}$$

The last equality comes from the fact that $Q \wedge \neg Q$ is always false.

(c)

$$\begin{aligned} & \neg(\exists x \text{ so that } (P(x) \wedge \neg Q(x)) \vee (P(x) \wedge R(x))) \\ & \Leftrightarrow \forall x, \neg((P(x) \wedge \neg Q(x)) \vee (P(x) \wedge R(x))) \\ & \Leftrightarrow \boxed{\forall x, \neg P(x) \vee (Q(x) \wedge \neg R(x))}. \end{aligned}$$

The last step is a modification of part (a).

(d)

$$\begin{aligned} & \neg(\forall x, \exists y \text{ so that } P(x, y) \vee \neg R(x, y)) \vee (\exists x \text{ so that } \forall y, \neg P(x, y) \wedge Q(x, y)) \\ & \Leftrightarrow (\exists x \text{ so that } \forall y, \neg P(x, y) \wedge R(x, y)) \vee (\exists x \text{ so that } \forall y, \neg P(x, y) \wedge Q(x, y)) \\ & \Leftrightarrow \boxed{\exists x \text{ so that } \forall y, \neg P(x, y) \vee (\neg R(x, y) \wedge Q(x, y))}. \end{aligned}$$

(e)

$$\begin{aligned} (P \Rightarrow Q) \wedge (R \vee Q) & \Leftrightarrow (\neg P \vee Q) \wedge (R \vee Q) \\ & \Leftrightarrow \boxed{(\neg P \wedge R) \vee Q}. \end{aligned}$$

(f) From problem 6(c), we see that this statement form is a tautology. Thus it is equivalent to any tautology we choose, say $\boxed{R \vee \neg R}$.

8. The blanks should be in order as follows:

$a \in \mathbb{Z}$;

$b \in \mathbb{Z}$;

$c \in \mathbb{Z}$;

$a \mid c$;

an integer;

$\frac{c}{b}$ is an integer;

$\frac{c}{b}$;

integer;

$a \mid c$.

9. The blanks should be in order as follows:

$$x \in \mathbb{R}^+;$$

$$x < n_1;$$

$$n_2;$$

$$n_1;$$

$$n_2;$$

$$\frac{1}{x},$$

$$\frac{1}{n_2},$$

$$\frac{1}{x},$$

$$\frac{1}{n} < x < n.$$