Math 220
Exam 1 Sample Problems
October 17, 2003

Included below are some sample problems for the exam. On the last page there is a list of axioms and propositions from the text and in class. You can use this list for your justifications in your proofs, and a copy of it (as is) will be provided with the exam.

1. Using only the axioms of the integers, prove the following.
   (a) For all a, b, $c \in \mathbb{Z}$, $(b - c)a = ab - ac$
   (b) For all a, $b \in \mathbb{Z}$, $a(-b) = -(ab)$.
   (c) For all $a \in \mathbb{Z}$, $(-1)a = -a$.
2. Let a, b, c, $d \in \mathbb{Z}$.
   (a) Suppose that $a < b$ and $c < d$. Then $a + c < b + d$.
   (b) Suppose that $ac < bc$ and $c > 0$. Prove that $a < b$.
   (c) Suppose that $ac < bc$ and $c < 0$. Prove that $a > b$.
3. Consider the statement: "For all integers x, if $x^2 - x - 6 = 0$, then $x = -2$ or $x = 3$."
   (a) What is the contrapositive of this statement?
   (b) What is its converse?
   (c) Which of the statement, its converse, and its contrapositive is true?
   (d) Prove or disprove the statement.
4. Prove or disprove the following statements:
   (a) For all integers x, $x^4 - x^2 = 0$ if and only if $x = 0$ or $x = 1$.
   (b) For all integers x, $9x^4 - x^2 = 0$ if and only if $x = 0$.
5. Consider the statement: "For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a is even or b is even."
   (a) What is the contrapositive of this statement?
   (b) What is its converse?
   (c) Which of the statement, its converse, and its contrapositive is true?
   (d) Prove or disprove the statement.
6. Prove by induction: $\displaystyle\sum_{i=2}^{n}(i^2 - 2i) = \frac{2n^3 - 3n^2 - 5n + 6}{6}$.

7. Prove the following by induction. Suppose $\{a_1, a_2, a_3, \ldots\}$ is an infinite sequence of integers such that $a_1 = 7$, $a_2 = 13$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for all integers $n \geq 3$. Then for all integers $n \geq 1$, $a_n = 5(3^{n-1}) - 2(-1)^n$.

8. Write the coefficient of x^8y^9 in $(2x - 3y)^{17}$.

9. Prove that $\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$.

10. Let $\mathbb{Z}^- = \{-n \mid n \in \mathbb{Z}^+\}$ be the set of negative integers.

    (a) Prove that -1 is the largest element of $\mathbb{Z}^-$.

    (b) Explain why every nonempty subset of $\mathbb{Z}^-$ has a largest element. (Formal proof not required.)

    (c) Prove that $\mathbb{Z}^-$ is closed under addition bot not closed under multiplication.

Known Axioms and Propositions

Axioms of $\mathbb{Z}$:

A0. Closure of $\mathbb{Z}$: $\mathbb{Z}$ is closed under $+$ and $\cdot$.
A1. Associtativity of $+$.
A2. Commutativity of $+$.
A3. 0 is the additive identity element.
A4. Every integer has an additive inverse.
A5. Associativity of $\cdot$.
A6. Commutativity of $\cdot$.
A7. 1 is the multiplicative identity element.
A8. Distributive law. (Include $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$.)
A9. Closure of $\mathbb{Z}^+$.
A10. Trichotomy law.
A11. Well-ordering principle.

Let a, b, $c \in \mathbb{Z}$.

P1. Subtracting from both sides: $a + b = a + c \Rightarrow b = c$.
P2. $a \cdot 0 = 0 \cdot a = 0$.
P3. $(-a)b = a(-b) = -(ab)$.
P4. $-(-a) = a$.
P5. $(-a)(-b) = ab$.
P6. $a(b - c) = ab - ac$.
P7. $(-1)a = -a$.
P8. $(-1)(-1) = 1$.

Let a, b, $c \in \mathbb{Z}$. Statements Q2–Q9 hold equally well if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Q1. Exactly one of the following holds $a < b$, $b < a$, or $a = b$.
Q2. $(a > 0 \Rightarrow -a < 0)$ and $(a < 0 \Rightarrow -a > 0)$.
Q3. If $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.
Q4. If $a > 0$ and $b < 0$, then $ab < 0$.
Q5. If $a < 0$ and $b < 0$, then $ab > 0$.
Q6. If $a < b$ and $b < c$, then $a < c$.
Q7. If $a < b$, then $a + c < b + c$.
Q8. If $a < b$ and $c > 0$, then $ac < bc$.
Q9. If $a < b$ and $c < 0$, then $ac > bc$.

Other statements we have proved (in homework or in class).

R1. $\forall a, b \in \mathbb{Z}$, $-(a + b) = -a - b$.
R2. $\forall a, b \in \mathbb{Z}$, if $ab = 0$, then $a = 0$ or $b = 0$.
R3. $\forall a, b, c \in \mathbb{Z}$, if $ab = ac$ and $a \neq 0$, then $b = c$.
R4. There is no integer x so that $0 < x < 1$.
R5. 1 is the smallest element of $\mathbb{Z}^+$.