Math 220
Exam 2 Sample Problems
Solutions Guide
October 21, 2003

1. (a) Let a, b, $c \in \mathbb{Z}$ be arbitrary. Then

$$\begin{aligned} (b-c)a &= a(b-c) && \text{(commutativity of } \cdot\text{)},\\ &= ab - ac && \text{(distributive laws)}. \end{aligned}$$

Therefore, the statement is true.

(b) Note that this is essentially the same as the proof of the first part of Prop. P3 on p. 153. Let a, $b \in \mathbb{Z}$ be arbitrary. Then if we can show that $a(-b)$ is the additive inverse of ab, we're done. That is, we want to show that $ab + a(-b) = 0$. So

$$\begin{aligned} ab + a(-b) &= a(b + (-b)) && \text{(distributive laws)},\\ &= a \cdot 0 && \text{(A4)}. \end{aligned}$$

One is tempted here to simply say that $a \cdot 0 = 0$, but this is not an axiom. Instead we need to supply that proof, which you should look up on pp.152–153.

(c) This we did in class. Let $a \in \mathbb{Z}$ be arbitrary. Then $a + (-1)a = (1 + (-1))a$ by the distributive laws. Since $1 + (-1) = 0$ by definition of additive inverses, we see that $a + (-1)a = 0 \cdot a$. Again, we need to appeal to the proof of $0 \cdot a = 0$.

2. (a) Let a, b, c, $d \in \mathbb{Z}$. Suppose that $a < b$ and $c < d$. By using Q7, we see that $a + c < b + c$ and that $c + b < d + b$. Combining these two inequalities (and using that $+$ is commutative), we see that $a + c < b + d$, which is what we wanted to prove.

(b) Let a, b, $c \in \mathbb{Z}$. Suppose that $ac < bc$ and $c > 0$. We'll use proof by contradiction. Suppose that $a \geq b$. Then since $c > 0$, we know that $ac \geq bc$, by Q8. But by our original choice, $ac < bc$. Having both $ac \geq bc$ and $ac < bc$ contradicts Q1. Thus our supposition that $a \geq b$ must be false, and so $a < b$.

(c) Similar to (b), but use Q9 instead of Q8.

3. (a) Contrapositive: For all integers x, if $x \neq -2$ and $x \neq 3$, then $x^2 - x - 6 \neq 0$.

(b) For all integers x, if $x = -2$ or $x = 3$, then $x^2 - x - 6 = 0$.

(c) All three are true.

(d) We have asserted that the statement is true. Proof: Let $x \in \mathbb{Z}$ be arbitrary. Assume that $x^2 - x - 6 = 0$. Note that

$$x^2 - x - 6 = (x-3)(x+2),$$

since by the distributive law, $(x-3)(x+2) = x(x+2) - 3(x+2) = x^2 + 2x - 3x - 6 = x^2 - x - 6$. Thus, since $x^2 - x - 6 = 0$, we conclude that

$$(x-3)(x+2) = 0.$$

By R2, we conclude that $x - 3 = 0$ or $x + 2 = 0$. Thus either $x = 3$ or $x = -2$ (by adding the 3 and -2 to each equation respectively).

4. (a) Our scratchwork tells us that the solutions of $x^4 - x^2 = 0$ are $x = 0, \pm 1$. Thus the statement is false. To disprove it, we prove its negation:

   There exists $x \in \mathbb{Z}$, so that either ($x^4 - x^2 = 0$ and $x \neq 0$ and $x \neq 1$), or ($x = 0$ or 1 and $x^4 - x^2 \neq 0$).

   (Why is this the negation?) This is a mouthful, and yet the proof is not hard. Proof of negation: Let $x = -1$. Then $x^4 - x^2 = (-1)^4 - (-1)^2 = 1 - 1 = 0$ (by P8 and A4). However $x \neq 0$ and $x \neq 1$. (We need only show one component of the OR statement true for our particular x, so we're done.)

   (b) Our scratchwork tells us that the solutions of $9x^4 - x^2 = 0$ are $x = 0, \pm\frac{1}{3}$. The only one of these which is an integer is $x = 0$, so in fact the statement is true. Proof: Let $x \in \mathbb{Z}$ be arbitrary. To prove the "if and only if," we prove both directions.

   ($\Rightarrow$): Suppose $9x^4 - x^2 = 0$. Then by the distributive law $x^2(9x^2 - 1) = 0$. By R2, it follows that either $x = 0$ or $9x^2 - 1 = 0$. However, if $9x^2 - 1 = 0$, then $9x^2 = 1$. But $x^2 \geq 1$ (by R5, since x^2 is a positive integer), so $9x^2 \geq 9 > 1$. Therefore, we can't have $9x^2 = 1$, and so it must be that $x = 0$.

   ($\Leftarrow$): Suppose that $x = 0$. Then $9x^4 - x^2 = 9 \cdot 0^4 - 0^2 = 0$. (By applications of P2.)

5. (a) Contrapositive: For all integers a, b, and c, if a is odd and b is odd, then $a^2 + b^2 \neq c^2$.

   (b) Converse: For all integers a, b, and c, if a is even or b is even, then $a^2 + b^2 = c^2$.

   (c) The statement and its contrapositive are true. The converse is false.

   (d) Proof: Let a, b, $c \in \mathbb{Z}$. Suppose that $a^2 + b^2 = c^2$. We want to show that either a or b is even. We'll use proof by contradiction: Assume that both a and b are odd.

   Then a^2 and b^2 are odd, and so $c^2 = a^2 + b^2$ is even. But if c^2 is even, it must be the case that c is even (otherwise, $c^2 = c \cdot c$ would be the product of two odd numbers and hence odd). So for some k, l, $m \in \mathbb{Z}$, we have $a = 2k + 1$, $b = 2l + 1$, and $c = 2m$. We substitute into the equation $a^2 + b^2 = c^2$ and obtain

$$(2k+1)^2 + (2l+1)^2 = (2m)^2.$$

   Expanding both sides and simplifying, we find:

$$4k^2 + 4k + 1 + 4l^2 + 4l + 1 = 4m^2$$
$$4k^2 + 4k + 4l^2 + 4l - 4m^2 = -2.$$

   Now the left-hand side is divisible by 4, but the right-hand side is not. This is a contradiction (since this is supposed to be an equality). Therefore, our assumption that a and b are both odd must be wrong, and so one of them must be even.

6. (The problem is supposed to tell you what n is, and in fact we are supposed to take n to be an integer ≥ 2.) Let the formula be represented by $P(n)$.

   Base case: (n=2). Then $\sum_{i=2}^{2}(i^2-2i) = 4-4 = 0$. The right-hand side is $\frac{1}{6}(2\cdot 2^3 - 3\cdot 2^2 - 5\cdot 2 + 6) = \frac{1}{6}(16-12-10+6) = 0$. So $P(2)$ is true.

   Induction step: Let k be an integer ≥ 2. Suppose that $P(k)$ is true. Therefore,

$$\sum_{i=2}^{k}(i^2-2i) = \frac{2k^3-3k^2-5k+6}{6}.$$

   Now then we have

$$\begin{aligned}\sum_{i=2}^{k+1}(i^2-2i) &= \left[\sum_{i=2}^{k}(i^2-2i)\right] + (k+1)^2 - 2(k+1)\\ &= \frac{2k^3-3k^2-5k+6}{6} + k^2+2k+1-2k-2\\ &= \frac{2k^3-3k^2-5k+6}{6} + \frac{6k^2-6}{6}\\ &= \frac{2k^3+3k^2-5k}{6}.\end{aligned}$$

   Hmmm, is this what we want? Let's see.... The right-hand side of $P(k+1)$ is

$$\begin{aligned}\frac{2(k+1)^3-3(k+1)^2-5(k+1)+6}{6} &= \frac{2(k^3+3k^2+3k+1)-3(k^2+2k+1)-5(k+1)+6}{6}\\ &= \frac{2k^3+6k^2+6k+2-3k^2-6k-3-5k-5+6}{6}\\ &= \frac{2k^3+3k^2-5k}{6}.\end{aligned}$$

   So the two sides of $P(k+1)$ agree, and the induction step is completed.

7. Let $P(n)$ be the statement $a_n = 5(3^{n-1}) - 2(-1)^n$. We want to show $P(n)$ is true for all integers $n \geq 1$.

   Base cases: ($n=1$ and $n=2$). Now $a_1 = 7$ is given. Also $5(3^{1-1}) - 2(-1)^1 = 5+2 = 7$. So $P(1)$ is true. Also, $a_2 = 13$ is given, and $5(3^{2-1}) - 2(-1)^2 = 15-2 = 13$. So $P(2)$ is true.

   Induction step: (Use 2nd Principle of Mathematical Induction) Let $k \in \mathbb{Z}^+$. Suppose that $P(i)$ is true for all $i \leq k$. We need to use this to show that $P(k+1)$ is true. Now by the definition of a_n, we see that $a_{k+1} = 2a_k + 3a_{k-1}$. Since $P(k)$ is true, we know that

$$a_k = 5(3^{k-1}) - 2(-1)^k.$$

   Since $P(k-1)$ is true, we have

$$a_{k-1} = 5(3^{k-2}) - 2(-1)^{k-1}.$$

Then we have

$$\begin{aligned} a_{k+1} &= 2a_k + 3a_{k-1} \\ &= 2(5(3^{k-1}) - 2(-1)^k) + 3(5(3^{k-2}) - 2(-1)^{k-1}) \\ &= 10 \cdot 3^{k-1} + 5 \cdot 3^{k-1} - 4(-1)^k - 6(-1)^{k-1} \\ &= 5 \cdot 3^k - 2(-1)^{k+1}. \end{aligned}$$

And this is what we want to prove, so $P(k+1)$ is true.

8. By the binomial theorem, the coefficient of x^8y^9 in $(2x-3y)^{17}$ is $2^8(-3)^9\binom{17}{8}$. This can be simplified, but there is no need to do so.

9. The binomial theorem says that

$$(-1+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}.$$

But of course $(-1+1)^n = 0^n$, and we are done.

10. (a) Proof by contradiction: Suppose that -1 is not largest in $\mathbb{Z}^-$, so there is an $a \in \mathbb{Z}^-$ with $a > -1$. Then $-a \in \mathbb{Z}^+$ by the definition of $\mathbb{Z}^-$, and multiplying the inequality by -1 (using Q9 of course), we see that

$$-a < 1.$$

But 1 is the smallest element of $\mathbb{Z}^+$ (by R5), so it can't be that $-a \in \mathbb{Z}^+$. This is a contradiction, so -1 is the largest element of $\mathbb{Z}^-$.

(b) If S is a nonempty subset of $\mathbb{Z}^-$, then the set $T = \{-n \mid n \in S\}$ is a nonempty subset of $\mathbb{Z}^+$. By the Well-Ordering Principle, T has a smallest element a. Since also $S = \{-m \mid m \in T\}$, we find that a is the largest element of S.

(c) Let $a, b \in \mathbb{Z}^-$. Then $a < 0$ and $b < 0$. So $a + b < 0$ (say using, problem 2(a) of this handout). So $a + b \in \mathbb{Z}^-$. However, $ab > 0$ by Q5, so $ab \notin \mathbb{Z}^-$.