Quiz #4 Solutions

1. Let a, b, and $c \in \mathbb{Z}$ be arbitrary. Assume that $a \neq 0$ and $ab = ac$. Then

$$\begin{aligned} ab - ac &= 0 && \text{(subtract } ac \text{ from both sides),} \\ a(b - c) &= 0 && \text{(distributive law).} \end{aligned}$$

   Since we know that $\forall x, y \in \mathbb{Z}$, if $xy = 0$, then $x = 0$ or $y = 0$, we conclude that either $a = 0$ or $b - c = 0$. Since $a \neq 0$ by assumption, we must have $b - c = 0$. Adding c to both sides, we see that $b = c$. Since a, b, and c were arbitrary, the statement is true for all choices of a, b, $c \in \mathbb{Z}$.

2. Let $x \in \mathbb{Z}$ be arbitrary. There are three cases to consider.

   Case 1: $x > 0$. If $x > 0$, then $x \geq 1$ (since 1 is the smallest positive integer by Corollary 5.1.7). Since $x > 0$, we can multiply the inequality $x \geq 1$ by x and obtain $x^2 \geq x$.

   Case 2: $x = 0$. If $x = 0$, then $x^2 = 0$, and clearly $0^2 \geq 0$.

   Case 3: $x < 0$. If $x < 0$, then $x^2 > 0$ (since the product of two negatives is positive). Combining these two inequalities, we see that

$$x < 0 \qquad \text{and} \qquad 0 < x^2,$$

   and so $x < x^2$. In particular, $x \leq x^2$.

   Thus in all cases, $x^2 \geq x$, which finishes the proof.