1. Suppose \( f(x, y) \) is a differentiable function such that it and its derivatives take on the following values:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
(x, y) & f(x, y) & f_x(x, y) & f_y(x, y) & f_{xx}(x, y) & f_{yy}(x, y) & f_{xy}(x, y) \\
\hline
(3, 6) & -44 & 0 & 0 & -12 & 0 & 12 \\
(3, 1) & 0 & 0 & -2 & -2 & -8 & 2 \\
(0, 0) & 10 & 0 & 0 & -12 & -6 & 0 \\
\hline
\end{array}
\]

Which of the following statements is true?

I. \( f(x, y) \) has a saddle point at \((x, y) = (3, 6)\).

II. \( f(x, y) \) has a local maximum at \((x, y) = (3, 1)\).

III. \( f(x, y) \) has a local maximum at \((x, y) = (0, 0)\).

(A) I only
(B) II only
(C) I and III only
(D) II and III only
(E) I, II, and III

Solution: \(\text{C} \)

At \((x, y) = (3, 6)\) we calculate \( D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 12 = -12 < 0 \). Therefore by the Second Derivative Test, \( f \) has a saddle point at \((3, 6)\) and I is true. At \((x, y) = (3, 1)\), we see that \( f_y(3, 1) \neq 0 \), so \( f \) does not have a critical point at \((3, 1)\) and thus cannot be a local maximum. So II is false. At \((x, y) = (0, 0)\), we calculate \( D = (-12)(-6) - 0 = 72 > 0 \) and \( f_x(0, 0) = -12 < 0 \), and so by the Second Derivative Test, \( f \) has a local maximum at \((0, 0)\). Therefore, III is true.

2. At what points in polar coordinates \((r, \theta)\) do the circle \( r = 6 \) and the cardioid \( r = 4 \cos \theta + 4 \) intersect?

(A) \((6, \frac{\pi}{3})\) and \((-6, -\frac{\pi}{3})\)
(B) \((6, \frac{\pi}{3})\) and \((-6, \frac{2\pi}{3})\)
(C) \((6, \frac{\pi}{6})\) and \((-6, -\frac{\pi}{6})\)
(D) \((6, \frac{\pi}{6})\) and \((-6, \frac{5\pi}{6})\)
(E) \((6, \frac{\pi}{4})\) and \((-6, -\frac{\pi}{4})\)

Solution: \(\text{B} \)

We solve the equation \( 4 \cos \theta + 4 = 6 \), and see that \( \cos \theta = \frac{1}{2} \). This has basic solutions \( \theta = \frac{\pi}{3} \) and \( \theta = -\frac{\pi}{3} \). Thus the answer is either A or B. It is tempting to pick A, but \((-6, \frac{2\pi}{3})\) points in the opposite direction from \( \theta = -\frac{\pi}{3} \), since the \( r \) value is negative. By the same reasoning, we see that \(\text{B} \) is the correct choice.
3. Suppose $Q$ is the rectangle in the $xy$-plane bounded by the lines $x = 0$, $x = 2$, $y = 0$, and $y = 3$, and suppose that $Q$ has a mass density given by

$$\delta(x, y) = xy^2.$$ 

If the total mass of $Q$ is $M$, which of the following integrals represents the $y$-coordinate of the center of mass of $Q$?

(A) $\frac{1}{M} \int_0^2 \int_0^3 x^2 y^2 \, dx \, dy$

(B) $\frac{1}{M} \int_0^2 \int_0^3 xy^3 \, dx \, dy$

(C) $\frac{1}{M} \int_0^3 \int_0^2 xy^3 \, dx \, dy$

(D) $\frac{1}{M} \int_0^2 \int_0^3 x^2 y^2 \, dy \, dx$

(E) $\frac{1}{M} \int_0^3 \int_0^2 xy \, dx \, dy$

Solution: [C] The formula for the $y$-coordinate of the center of mass is

$$\overline{y} = \frac{1}{M} \int \int_R y\delta(x, y) \, dA,$$

where $R$ is the rectangle described in the problem. Thus the correct answer is either B or C. Checking the limits of integration, together with the orders of integration given, we see that [C] is the answer.

4. Let $R$ be the region in the $xy$-plane bounded by the graphs of $y = 2x$ and $y = \sqrt{8x}$. Which of the following iterated integrals represents $\int \int_R f(x, y) \, dA$?

(A) $\int_0^4 \int_{2x}^{\sqrt{8x}} f(x, y) \, dy \, dx$

(B) $\int_0^2 \int_{2x}^{\sqrt{8x}} f(x, y) \, dy \, dx$

(C) $\int_0^4 \int_{\sqrt{8x}}^{2x} f(x, y) \, dy \, dx$

(D) $\int_0^2 \int_{y^2/8}^{y^2/2} f(x, y) \, dx \, dy$

(E) $\int_0^4 \int_{y^2/8}^{y^2/2} f(x, y) \, dx \, dy$

Solution: [E] First we find the points of intersection of the two curves, and setting $2x = \sqrt{8x}$, we see that $4x^2 = 8x$, and thus

$$4x^2 - 8x = 4x(x - 2) = 0.$$
The line and the curve intersect at \((0, 0)\) and \((2, 4)\). The region \(R\) is thus bounded on the left by a parabola pointing to the right and on the right by a line. Solving for \(x\) we see that the parabola is \(x = y^2/8\) and the line is \(x = y/2\), which leads to the solution in E.

5. If we change the order of integration of the integral \(\int_0^2 \int_0^{4-x^2} f(x, y) \, dy \, dx\), we obtain which integral below?

(A) \(\int_0^2 \int_0^{\sqrt{y-4}} f(x, y) \, dx \, dy\)

(B) \(\int_0^4 \int_0^{\sqrt{4-y}} f(x, y) \, dx \, dy\)

(C) \(\int_0^2 \int_0^4 f(x, y) \, dx \, dy\)

(D) \(\int_0^4 \int_0^4 f(x, y) \, dx \, dy\)

(E) None of the above

Solution: B We first need to identify the region of integration, which is defined by the inequalities

\[ R = \{(x, y) \mid 0 \leq x \leq 2, \ 0 \leq y \leq 4 - x^2\}. \]

Thus \(R\) is the region in the first quadrant bounded by a parabola facing downward with vertex at \((0, 4)\) and the coordinate axes. Switching the order of the inequalities, we start with \(0 \leq y \leq 4\), and solving for \(x\) in the equation of the parabola, we have \(x = \sqrt{4-y}\). For fixed \(y\), we are taking \(x\)-values extending from the \(y\)-axis to the parabola, and so \(0 \leq x \leq \sqrt{4-y}\). Thus the answer is B.

6. Let \(S\) be the surface defined by the graph of the paraboloid \(z = x^2 + y^2\) lying above the triangle in the \(xy\)-plane with vertices \((0, 1)\), \((0, 7)\), and \((2, 7)\). Write down an iterated double integral that expresses the surface area of \(S\). You do not need to evaluate the integral.

Solution: The triangle in question is bounded on the left by the \(y\)-axis, on the top by the line \(y = 7\), and on the right/bottom by the line \(y = 3x + 1\). Therefore, the triangle is defined by the inequalities \(0 \leq x \leq 2\) and \(3x + 1 \leq y \leq 7\). Now for the surface area of \(S\), we have

\[ \text{S.A.} = \int \int_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA = \int_0^2 \int_{3x+1}^7 \sqrt{4x^2 + 4y^2 + 1} \, dy \, dx. \]

Equally good is to have changed the order of the variables and found

\[ \text{S.A.} = \int_1^7 \int_0^{(y-1)/3} \sqrt{4x^2 + 4y^2 + 1} \, dx \, dy. \]
7. Let \( f(x, y) = 4x^2 - 4xy + y^2 \). What are the maximum and minimum values of \( f(x, y) \) on the circle \( x^2 + y^2 = 20 \)? At what points \((x, y)\) do they occur? You must use the method of Lagrange multipliers.

**Solution:** Let \( g(x, y) = x^2 + y^2 - 20 \). By the method of Lagrange multipliers we want to solve the system of equations,

\[
\begin{align*}
g(x, y) &= 0, \\
f_x &= \lambda g_x, \\
f_y &= \lambda g_y,
\end{align*}
\]

We then have the three equations

\[
\begin{align*}
x^2 + y^2 &= 20 \quad (1) \\
8x - 4y &= 2\lambda x \quad (2) \\
-4x + 2y &= 2\lambda y. \quad (3)
\end{align*}
\]

If we multiply (3) by 2, we obtain \(-8x + 4y = 4\lambda y\), and if we add this to (2), we obtain

\[
0 = 2\lambda x + 4\lambda y = 2\lambda(x + 2y).
\]

Thus either \(\lambda = 0\) or \(x = -2y\). If \(\lambda = 0\), then equation (2) reduces to \(8x - 4y = 0\) or \(y = 2x\). Substituting this into (1), we find that \(5x^2 = 20\), and so \(x = \pm 2\). Since \(y = 2x\), we find two solutions:

\[
(x, y) = (2, 4), \quad (x, y) = (-2, -4).
\]

Now if \(x = -2y\), then we substitute into (1) and find \(5y^2 = 20\). Thus \(y = \pm 2\), and using \(x = -2y\), we find two more solutions:

\[
(x, y) = (-4, 2), \quad (x, y) = (4, -2).
\]

To find the maximum and minimum values, we evaluate \(f\) at these four points:

\[
\begin{align*}
f(2, 4) &= 0 \\
f(-2, -4) &= 0 \\
f(-4, 2) &= 100 \\
f(4, -2) &= 100.
\end{align*}
\]

Thus the **maximum value is 100**, occurring at \((x, y) = (-4, 2)\) or \((4, -2)\). Furthermore, the **minimum value is 0**, occurring at \((x, y) = (2, 4)\) or \((-2, -4)\).

8. Let \(E\) be the solid in 3-space which is bounded above by the surface \(z = \sqrt{27 - 2x^2 - 2y^2}\) (the top half of an ellipsoid) and below by the plane \(z = 3\).

(a) Express the volume of \(E\) as an iterated double integral in rectangular coordinates. (You do not need to evaluate the integral.)

(b) Convert the integral from part (a) into an iterated double integral in polar coordinates.
(c) Evaluate the integral from part (b) to find the volume of $E$.

**Solution:** We first need to identify the region $R$ in the $xy$-plane sitting below $E$. We know that $E$ is bounded above by a paraboloid and below by a plane, so the boundary of $R$ will be the projection of the intersection of the two surfaces onto the $xy$-plane. The intersection is obtained by solving

$$3 = \sqrt{27 - 2x^2 - 2y^2},$$

which becomes $9 = 27 - 2x^2 - 2y^2$. This simplifies to $x^2 + y^2 = 9$, and thus $R$ is the circle of radius 3 centered at the origin.

(a) For the volume in rectangular coordinates, we have

$$V = \int_R \left( \sqrt{27 - 2x^2 - 2y^2} - 3 \right) dA = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left( \sqrt{27 - 2x^2 - 2y^2} - 3 \right) dy \, dx.$$

(b) In polar coordinates, we have

$$V = \int_R \left( \sqrt{27 - 2x^2 - 2y^2} - 3 \right) dA = \int_0^{2\pi} \int_0^3 \left( \sqrt{27 - 2r^2} - 3 \right) r \, dr \, d\theta.$$

(c) To evaluate the integral in (b) we find

$$V = 2\pi \int_0^3 r \sqrt{27 - 2r^2} \, dr - 2\pi \int_0^3 3r \, dr$$

$$= -\frac{2\pi}{4} \int_{27}^{9} \sqrt{u} \, du - 2\pi \left[ \frac{3}{2} r^2 \right]_0^3$$

(using $u = 27 - 2r^2$, $du = -4r \, dr$)

$$= \frac{\pi}{2} \left[ \frac{3}{2} u^{3/2} \right]_{27}^{9} - 2\pi \cdot \frac{27}{2}$$

$$= \frac{\pi}{3} \left[ 81\sqrt{3} - 27 \right] - 27\pi$$

$$= \left[ 27\sqrt{3} - 36 \right] \pi.$$

9. [14 points; (a) 6 pts., (b) 8 pts.] Consider the integral

$$\int_0^4 \int_{y/2}^{2} (x^2 + 1)^{1/3} \, dx \, dy.$$

(a) Describe the region of integration in the $xy$-plane for the integral above. Either describe in words, or draw a labeled sketch of the region.

(b) Evaluate the integral.

**Solution:** (a) The region is the triangle bounded by the lines $y = 2x$, $y = 0$, and $x = 2$. Its vertices are $(0,0)$, $(2,0)$, and $(2,4)$. Various answers were appropriate.
(b) In order to perform the integration it is necessary to reverse the order of the variables. We see that

\[
\int_0^4 \int_{y/2}^{2} (x^2 + 1)^{1/3} \, dx \, dy = \int_0^2 \int_0^{2x} (x^2 + 1)^{1/3} \, dy \, dx
\]

\[
= \int_0^2 2x(x^2 + 1)^{1/3} \, dx
\]

\[
= \int_1^5 u^{1/3} \, du \quad \text{(using } u = x^2 + 1, \, du = 2x \, dx)\]

\[
= \left[ \frac{3}{4} u^{4/3} \right]_1^5
\]

\[
= \frac{3}{4} \cdot 5^{4/3} - \frac{3}{4}
\]

\[
= \frac{3}{4} \left( 5^{4/3} - 1 \right).
\]