1. Since $\rho^2 = z^2 + r^2$, we see that $\rho^2 = 4 + 12 = 16$, and so $\rho = 4$. Also $z = \rho \cos \phi$, so $\cos \phi = \frac{2}{4} = \frac{1}{2}$, which implies that $\phi = \frac{\pi}{3}$. Finally, $\theta$ is unchanged between cylindrical and spherical coordinates, so the answer is $\{(\rho, \theta, \phi) = (4, \frac{\pi}{3}, \frac{\pi}{3})\}$.

2. By setting $5 = 9 - x^2 - y^2$, we see that the curve of intersection projects onto the circle $x^2 + y^2 = 4$ in the $xy$-plane. In the first quadrant (which sits below the first octant) this projects down to the quarter disk $0 \leq y \leq 2$ and $0 \leq x \leq \sqrt{4 - y^2}$, which then gives the bounds for $x$- and $y$-coordinates of points in $E$. Finally the $z$-coordinate of a point in $E$ is bounded by $5 \leq z \leq 9 - x^2 - y^2$, so the volume of $E$ is

$$\text{Vol}(E) = \iiint_{E} dV = \int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{\sqrt{18}} d\rho \sin \phi \rho^2 d\rho d\phi d\theta.$$ 

3. The region $E$ is the snow cone shaped region between the cone and the sphere.

(a) We calculate the bounds on $E$ in spherical coordinates. First $0 \leq \theta \leq 2\pi$, since we are rotating completely around the $z$-axis. The cone $z^2 = x^2 + y^2$ determines the range of $\phi$. Since $z = \rho \cos \phi$ and $r = \rho \sin \phi$, we see that on this cone, where $z^2 = r^2$, we have $\tan \phi = 1$, which means the cone is described by the equation

$$\phi = \frac{\pi}{4}.$$ 

Thus the $\phi$ range of $E$ is $0 \leq \phi \leq \frac{\pi}{4}$. Finally, $\rho$ ranges from 0 to the radius of the sphere, $\sqrt{18}$. Thus

$$\text{Vol}(E) = \iiint_{E} dV = \int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{\sqrt{18}} \rho^2 \sin \phi d\rho d\phi d\theta.$$ 

(b) To convert the limits of integration on $E$ into cylindrical coordinates, we first see that $E$ is contained in the cylinder $x^2 + y^2 = 9$ (obtained by substituting $z^2 = x^2 + y^2$ into $x^2 + y^2 + z^2 = 18$). Thus $E$ sits over the disk of radius 3 centered at the origin and is bounded above by the sphere and below by the cone. In cylindrical coordinates, the sphere is given by $z^2 + r^2 = 18$ and the cone is given by $z = r$. The mass density is given by $d = x^2 + y^2 = r^2$. All together then, the mass of $E$ is

$$\text{mass}(E) = \iiint_{E} dV = \int_{0}^{2\pi} \int_{0}^{3} \int_{r}^{\sqrt{18-r^2}} r^3 dz dr d\theta.$$ 

4. After a short computation, you find that $\text{div}(\nabla f) = -z(x^2 + y^2) \sin(xy)$. 


5. Since \( dx = 3 \, dt \) and \( dy = 4 \, t \, dt \), we see that
\[
\int_C 3y \, dx - x^2 \, dy = \int_0^1 \left( 3(2t^2) \cdot 3 - 9t^2(4t) \right) \, dt = \int_0^1 (18t^2 - 36t^3) \, dt = -3.
\]

6. From the hint we look for a potential function \( f(x, y, z) \) for \( \mathbf{F} \), and we see that we can take \( f(x, y, z) = x + y + z + xyz \) so that \( \nabla f = \mathbf{F} \). Since the vector field is conservative, we then need to calculate \( f \) at the endpoints of \( C \) and subtract. When \( t = 0 \), we see that the initial point of \( C \) is \((0, 0, 0)\), and when \( t = \pi \), the final point of \( C \) is \((\pi, 0, \pi e)\). Thus
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi, 0, \pi e) - f(0, 0, 0) = \pi + \pi e.
\]

7. We use the Divergence Theorem. First we see that \( \text{div}(\mathbf{F}) = 4 \). If we let \( E \) denote the solid region surrounded by \( S \) and the unit disk \( D \) in the \( xy \)-plane, then by the Divergence Theorem
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \text{div}(\mathbf{F}) \, dV - \iint_D \mathbf{F} \cdot \mathbf{n} \, dS
\]
\[
= \iiint_E 4 \, dV - \iint_D \langle x, y, 2z \rangle \cdot \langle 0, 0, -1 \rangle \, dS.
\]
The triple integral evaluates to \( 4 \) times the volume of \( E \): check that \( \iiint_E 4 \, dV = 2\pi \). The surface integral over the disk \( D \) is 0, because at every point in \( D \) we have \( z = 0 \), so the dot product in the integrand is always 0 on \( D \). Thus the answer is \( \frac{2\pi}{2} \).

8. (a) Using that \( ds = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt \), we see
\[
\int_C (x^2 + y^2 + z^2) \, ds = \int_0^{2\pi} \left( 16 \cos^2 t + 16 \sin^2 t + 9t^2 \right) \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} \, dt.
\]
Evaluating the integral on the right, we obtain an answer of \( 160\pi + 120\pi^3 \).

(b) Let \( R \) be the triangular region bounded by the points \((0, 0)\), \((2, 0)\), and \((2, 4)\) in the \( xy \)-plane. Then Green’s Theorem says that
\[
\int_C (3ye^{x^2} + 2e^x) \, dx + \left( \frac{3}{2} x^2 + \sin y \right) = \iint_R (3x - 3e^{x^2}) \, dA,
\]
\[
= \int_0^2 \int_0^{2x} (3x - 3e^{x^2}) \, dy \, dx.
\]
Evaluating the iterated integral, we obtain an answer of \( 19 - 3e^4 \).

9. (a) \[
\int_{-1}^1 \int_{\sqrt{1-x^2}}^{-\sqrt{1-x^2}} -4 \, dy \, dx
\]
(b) Use the usual parametrization of $C : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$. So $dx = -\sin t \, dt$, $dy = \cos t \, dt$. Then

$$\oint F \cdot dr = \int_C (2y \, dx - 2x \, dy) = \int_0^{2\pi} (-2 \sin^2 t - 2 \cos^2 t) \, dt.$$ 

This integral evaluates to $-4\pi$.

(c) The integral in (a) is simply $-4$ times the area of the region of integration, which is a disk of radius 1. Thus,

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -4 \, dy \, dx = -4\pi$$

as expected from (b).

10. By Stokes’ Theorem, the surface integral in question is the same as a line integral along the boundary of $S$. More to the point, if we let $C$ circle of radius 3 in the $xy$-plane defined by $x = 3 \cos t, y = 3 \sin t, z = 0, 0 \leq t \leq 2\pi$, then

$$\iint_S (\text{curl } F) \cdot \mathbf{n} \, dS = \oint_C F \cdot dr = \int_0^{2\pi} (-9 \sin^2 t - 9 \cos^2 t) \, dt = -18\pi.$$ 

11. In this problem we use Stokes’ Theorem to evaluate the desired line integral in terms of a surface integral. Given that $F = \langle y, -x, yz \rangle$, we calculate that

$$\text{curl } F = \langle z, 0, -2 \rangle.$$ 

Now if we let $\mathbf{n}$ be the upward unit normal vector on $S$, then

$$\oint F \cdot dr = \iint_S (\text{curl } F) \cdot \mathbf{n} \, dS.$$ 

When calculating surface integrals on surfaces defined by the graph of a function, in this case $z = x^2 + y^2$, we have some relatively nice formulas. For example, see formula (8) on p. 922 of Stewart. Then if we let $D$ be the unit disk in the $xy$-plane,

$$\iint_S (\text{curl } F) \cdot \mathbf{n} \, dS = \iint_S \langle z, 0, -2 \rangle \cdot \mathbf{n} \, dS = \iint_D (-z(2x) - 0 \cdot (2y) - 2) \, dA.$$ 

The double integral on the right then evaluates to $-2\pi$. 

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