MATH 251.509
Final Examination
May 10, 2006

NAME ______________________________
SIGNATURE ___________________________

This exam consists of **15** problems, numbered **1–15**. For partial credit you must present your work clearly and understandably and justify your answers.

The use of calculators is not permitted on this exam.

The point value for each question is shown next to each question.

CHECK THIS EXAMINATION BOOKLET BEFORE YOU START. THERE SHOULD BE 15 PROBLEMS ON 9 PAGES (INCLUDING THIS ONE).

Do not mark in the box below.

	Points Possible	Credit
1–10	70	
11	16	
12	18	
13	18	
14	18	
15	10	
Total	150	

Multiple Choice: [7 points each] In each of Problems 1–10, circle the best answer.

1. The function $f(x, y) = x^2y^2 - 4x - y$ has a critical point at $(x, y) = \left(\frac{1}{2}, 2\right)$. According to the Second Derivative Test, this point is

(A) a local minimum.

(B) a local maximum.

(C) a saddle point.

(D) an inflection point.

(E) none of the above. The Second Derivative Test **fails**.

2. Find the area of the parallelogram with vertices $(0, 0, 0)$, $(2, 1, 1)$, $(3, 4, 1)$, and $(1, 3, 0)$.

(A) $\sqrt{35}$

(B) $\sqrt{59}$

(C) 35

(D) 59

(E) 5

3. Calculate $\iint_R \frac{1}{x^2 + y^2}\, dA$, where R is the ring-shaped region in the xy-plane between the circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

(A) 7π

(B) 2π

(C) $\frac{\pi}{6}$

(D) $2\pi \ln 4$

(E) $2\pi \ln \frac{4}{3}$

4. Suppose $z = \dfrac{x}{y}$, where $x = x(r,s)$ and $y = y(r,s)$ are functions satisfying

$$x(1,2) = 3 \qquad \frac{\partial x}{\partial r}(1,2) = 5 \qquad \frac{\partial x}{\partial s}(1,2) = 7$$

$$y(1,2) = 4 \qquad \frac{\partial y}{\partial r}(1,2) = 6 \qquad \frac{\partial y}{\partial s}(1,2) = 8.$$

Find $\dfrac{\partial z}{\partial r}$ at $(r,s) = (1,2)$.

(A) 1

(B) $\dfrac{1}{8}$

(C) $\dfrac{1}{4}$

(D) $\dfrac{3}{4}$

(E) $-\dfrac{1}{16}$

5. Let $f(x,y,z) = z\sin(xy)$. Find $\operatorname{div}(\nabla f)$.

(A) $z(x+y)\cos(xy) + \sin(xy)$

(B) $-z(x^2+y^2)\sin(xy)$

(C) $\langle yz\cos(xy), xz\cos(xy), \sin(xy)\rangle$

(D) $\langle -y^2z\sin(xy), -x^2z\sin(xy), 0\rangle$

(E) 0

6. Let L be the line given by $x = 1+4t$, $y = -2+3t$, $z = (5\sqrt{3})t$. Let $\mathbf{n}$ be a normal vector to the plane $4x+3y-(5\sqrt{3})z = 12$. What is the acute angle between L and $\mathbf{n}$?

(A) 0

(B) $\dfrac{\pi}{6}$

(C) $\dfrac{\pi}{4}$

(D) $\dfrac{\pi}{3}$

(E) $\dfrac{\pi}{2}$

7. Let C be the curve in the xy-plane given by $x = 3t$, $y = 2t^2$, $0 \leq t \leq 1$. Evaluate $\displaystyle\int_C 3y\,dx - x^2\,dy$.

(A) -3

(B) -2

(C) 0

(D) 1

(E) 2

8. Let $f(x, y) = e^{x-y} + \sqrt{x^2 + y^2}$. Consider the following statements:

I. The domain of $f(x, y)$ is the set $\{(x, y) \mid x \neq y\}$.
II. The range of $f(x, y)$ is the set $\{z \mid z \geq 0\}$.
III. $f(x, y)$ is differentiable at $(x, y) = (0, 0)$.

Which of the above statements is true?

(A) I only

(B) II only

(C) III only

(D) II and III only

(E) None are true.

9. The value of $\displaystyle\int_0^{2\pi} \int_0^3 \int_1^3 r\,dz\,dr\,d\theta$ is the volume of which solid below?

(A) a right circular cone of base radius 2 and height 3

(B) a right circular cone of base radius 3 and height 2

(C) a right circular cylinder of radius 2 and height 3

(D) a right circular cylinder of radius 3 and height 2

(E) a hemisphere of radius 3

10. Let D be the disk of radius 2 centered at the origin in the xy-plane. Let R be the part of D in the second quadrant. The x-coordinate of the centroid of R is given by which integral below?

(A) $\displaystyle \frac{1}{4\pi}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx$

(B) $\displaystyle \frac{1}{\pi}\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} x\,dx\,dy$

(C) $\displaystyle \frac{1}{\pi}\int_{0}^{2}\int_{-\sqrt{4-y^2}}^{0} x\,dx\,dy$

(D) $\displaystyle \frac{1}{\pi}\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} y\,dy\,dx$

(E) $\displaystyle \frac{1}{\pi}\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} x\,dy\,dx$

11. [16 points] Evaluate $\displaystyle \int_{0}^{4}\int_{\sqrt{y}}^{2} e^{x^3+1}\,dx\,dy$.

12. [18 points; (a) 2 pts., (b) & (c) 8 pts. each] Consider the two surfaces in 3-space,

$$S_1 : f(x, y, z) = 2x^2 + y^2 + z^2 - 10 = 0,$$
$$S_2 : g(x, y, z) = x^2 - z + 1 = 0.$$

(a) Verify that the point $P = (1, 2, 2)$ is contained in both surfaces.

(b) Find the equation of the tangent plane to S_1 at the point P.

(c) The tangent plane to S_1 at P and the tangent plane to S_2 at P intersect in a line. What are the parametric equations of that line?

13. [18 points] Use a triple integral in spherical coordinates to find the volume of the solid within the sphere $x^2 + y^2 + z^2 = 16$, outside the cone $z = \sqrt{x^2 + y^2}$, and above the xy-plane.

14. [18 points] Let $\mathbf{F} = 2y\mathbf{i} - 2x\mathbf{j}$ be a vector field in the xy-plane. Let C be the unit circle oriented counterclockwise, and let R be the closed unit disk.

(a) According to Green's Theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is equal to what double integral?

(b) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.

(c) Evaluate the double integral you provided in part (a) directly.

15. [10 points] **Recall:** Suppose $f(x, y)$ is a differentiable function defined on a region R in the xy-plane. Let S be the surface that is the graph in 3-space of f above R, and let $\mathbf{n}$ be an upward unit normal vector to S. If $\mathbf{G}$ is any vector field in 3-space, then the flux of $\mathbf{G}$ across S is given by

$$\iint_S \mathbf{G} \cdot \mathbf{n}\, dS = \iint_R \mathbf{G} \cdot (-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k})\, dA.$$

The problem: Let S be the graph of $f(x, y) = x^2 + y^2$ above the unit disk R in the xy-plane. The boundary of S is the circle C that is a translate of the usual unit circle in the xy-plane, 1 unit directly upward.

Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + yz\mathbf{k}$. Use Stokes' Theorem to evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is oriented counter-clockwise.

(Hint: Stokes' Theorem says $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S$ ______ $\cdot\, \mathbf{n}\, dS$.)