**Math 304**
**Exam 1 Sample Problems Solutions**
February 17, 2004

These are the answers for some of the sample problems. However, on many problems you will need to show more work to get full credit.

1. For the first system of equations:

   (a) $A = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$; and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$.

   (b) After performing the row reductions (which you need to write out), the reduced row echelon form is
   $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

   (c) The set of solutions is $\{(s, t, u) = (5, -9, 4)\}$.

   For the second system of equations:

   (a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ v \\ w \end{bmatrix}$; and $\mathbf{b} = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}$.

   (b) After performing the row reductions (which you need to write out), the reduced row echelon form is
   $$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & -6 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 10 \end{array}\right].$$

   (c) The set of solutions is $\{(-6 - \alpha + \beta, \alpha, 3 - \beta, 10 - \beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}$.

2. Take the determinants by expanding by minors along a row or column:

   (a) Determinant $= 0$; not invertible (since the determinant is 0).

   (b) Determinant $= 1$; invertible.

   (c) Determinant $= -1$; invertible.

3. Only the matrices in (b) and (c) are invertible. Find inverse by row reducing $[A \mid I]$ into reduced row eceholon form, namely $[I \mid A]$.

(b) Inverse $= \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

(c) Inverse $= \begin{bmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{bmatrix}$

4. (a) Let $E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Answers may vary. Can take $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$; then $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; then $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$; then $E_4 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; then $E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

5. Here we first calculate

$$AB = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, $(AB)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, and similarly $(AB)^3 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$. In general,

$$(AB)^n = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}.$$

6. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and so $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Thus,

$$\det(A) = \det(A^T) = ad - bc.$$

Now since $AA^T = I$, we have that $A^T = A^{-1}$. Since

$$\det(A^{-1}) = \frac{1}{\det(A)},$$

we see that

$$\det(A) = \det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)};$$

that is $\det(A) = \frac{1}{\det(A)}$, which gives $\det(A)^2 = 1$. Thus, $\det(A) = \pm 1$.

7. The lower left entry of $A$ should be $-\cos(x)\sin(y)$. Expand the first row by minors:

$$\begin{aligned}
\det(A) &= \cos(y)\begin{vmatrix} \cos(x) & -\sin(x)\cos(y) \\ \sin(x) & \cos(x)\cos(y) \end{vmatrix} - 0 + \sin(y)\begin{vmatrix} \sin(x)\sin(y) & \cos(x) \\ -\cos(x)\sin(y) & \sin(x) \end{vmatrix} \\
&= \cos(y)(\cos^2(x)\cos(y) + \sin^2(x)\cos(y)) + \sin(y)(\sin^2(x)\sin(y) + \cos^2(x)\sin(y)) \\
&= \cos^2(y)(\cos^2(x) + \sin^2(x)) + \sin^2(y)(\sin^2(x) + \cos^2(x)) \\
&= \cos^2(y)\cdot 1 + \sin^2(y)\cdot 1 \\
&= 1.
\end{aligned}$$

So $\det(A) = 1$, regardless of $x$ and $y$.