

1. [18 points] Consider the system of equations

$$\begin{aligned}w + 2y - 3z &= -10 \\2w - 2x + 2y - 6z &= -28 \\-3w - 6y + 10z &= 33\end{aligned}$$

(a) Write down the augmented matrix for this system of equations with the variables ordered w, x, y, z .

Solution:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -10 \\ 2 & -2 & 2 & -6 & -28 \\ -3 & 0 & -6 & 10 & 33 \end{array} \right]$$

(b) Carry out on paper, step by step, the row reduction algorithm on the augmented matrix and transform it into reduced row echelon form.

Solution: This is fairly straightforward, and you eventually arrive in the form

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right]$$

(c) Describe completely the set of solutions of this system of equations.

Solution: Once we have our equations in reduced row echelon form as in (b), our original system is equivalent to the system

$$\begin{aligned}w &= -1 - 2y \\x &= 4 - y \\z &= 3.\end{aligned}$$

Thus the set of solutions is

$$\{(-1 - 2y, 4 - y, y, 3) \mid y \in \mathbb{R}\}.$$

2. [18 points] Consider the matrix

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

Find the inverse matrix of A . Carry this out on paper, step by step.

Solution: You find the inverse by reducing the matrix $[A|I]$ to the form $[I|B]$. The matrix B is then the inverse of A . Thus you row reduce

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & -2 & -1 & 0 & 0 & 1 \end{array} \right].$$

The inverse of A is then

$$A^{-1} = \begin{bmatrix} 2 & -3 & 2 \\ -1 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix}.$$

3. [15 points] For each statement below, write down whether the statement is **True** or **False**.

(a) Suppose $A = E_1 E_2 \cdots E_k$, where E_1, \dots, E_k are elementary matrices. Then A must be invertible.

True. Elementary matrices are invertible, and the product of invertible matrices is invertible.

(b) For square matrices A and B , it must be true that $(A^T B^T)^T = AB$.

False. In fact, $(A^T B^T)^T = BA$.

(c) For 2×2 matrices A and B , we must have $\det(A + B) = \det(A) + \det(B)$.

False. Easy to find a counterexample.

(d) For a square matrix A , if a linear system of equations $A\mathbf{x} = \mathbf{b}$ has a solution then A must be invertible.

False. If there is more than one solution, then A will not be invertible. A is invertible precisely when there is a *unique* solution.

(e) For square matrices A , B , and C , if $AB = C$ and also C is non-singular, then both A and B must be non-singular.

True. Take the determinants. $\det(C) = \det(AB) = \det(A)\det(B)$. Since C is non-singular, $\det(C) \neq 0$. Therefore, $\det(A) \neq 0$ and $\det(B) \neq 0$.

4. [12 points] Consider the matrix

$$B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 1 & -1 \\ 2 & 3 & -5 \end{bmatrix}.$$

- (a) Compute the determinant of B . Show your work.

Solution: Expand by minors, say along the top row:

$$\begin{aligned} \det(B) &= 0 \cdot \begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & -1 \\ 2 & -5 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ &= 0 + (-3) + 3 \\ &= 0. \end{aligned}$$

- (b) Is B invertible? Why or why not?

Solution: No, B is not invertible because its determinant is 0.

- (c) Is B row equivalent to the identity matrix? Why or why not?

Solution: No, if B were row equivalent to the identity matrix, then it would be invertible.

5. [10 points] Let B be given as in Problem #4 above. Let

$$C = \begin{bmatrix} 0 & -3 & 9 \\ 1 & 1 & -1 \\ 2 & 6 & -14 \end{bmatrix}.$$

Find elementary matrices E and F so that $EFB = C$.

Solution: There are essentially two correct answers to this problem:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{with} \quad F = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$E = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

6. [12 points] Let B be an $n \times n$ matrix; let I be the $n \times n$ identity matrix; and let O be the $n \times n$ zero matrix. Let

$$A = \left[\begin{array}{c|c} O & 2I \\ \hline I & B \end{array} \right].$$

The inverse matrix of A has the form

$$A^{-1} = \left[\begin{array}{c|c} E & F \\ \hline G & H \end{array} \right],$$

for $n \times n$ matrices E , F , G , and H . What are E , F , G , and H in terms of B , I , and O ?

Solution: We calculate

$$\left[\begin{array}{c|c} O & 2I \\ \hline I & B \end{array} \right] \left[\begin{array}{c|c} E & F \\ \hline G & H \end{array} \right] = \left[\begin{array}{c|c} 2G & 2H \\ \hline E + BG & F + BH \end{array} \right]$$

If the right-hand matrix is supposed to be the identity $\left[\begin{array}{c|c} I & O \\ \hline O & I \end{array} \right]$, then we have

$$\begin{aligned} 2G &= I \\ 2H &= O \\ E + BG &= O \\ F + BH &= I. \end{aligned}$$

From this we conclude that

$$\begin{aligned} H &= O \\ F &= I \\ G &= \frac{1}{2}I \\ E &= -\frac{1}{2}B \end{aligned}$$

7. [15 points] At the start of Year 1 we begin observing a population of rabbits that has $M_1 = 5000$ mature rabbits and $Y_1 = 1000$ young rabbits. The population changes every year according to the following rules: At the start of each year,
- Every young rabbit from the previous year becomes a mature one;
 - There are new young rabbits totalling 10% of the previous year's mature rabbits;
 - 10% of all mature rabbits from the previous year die off.

(a) How many mature rabbits and how many young rabbits will there be at the start of Year 2? Year 3?

Solution:

Year 2: 500 young rabbits; 5500 mature rabbits.

Year 3: 550 young rabbits; 5450 mature rabbits.

(b) If at the start of Year n there are M_n mature rabbits and Y_n young rabbits, how many of each will there be at the start of the next year? That is, what are M_{n+1} and Y_{n+1} in terms of M_n and Y_n ?

Solution:

$$\begin{aligned}M_{n+1} &= M_n + Y_n - \frac{1}{10}M_n \\Y_{n+1} &= \frac{1}{10}M_n\end{aligned}$$

(c) Find a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $\begin{bmatrix} M_{n+1} \\ Y_{n+1} \end{bmatrix} = A \begin{bmatrix} M_n \\ Y_n \end{bmatrix}$.

Solution: $A = \begin{bmatrix} \frac{9}{10} & 1 \\ \frac{1}{10} & 0 \end{bmatrix}$.

(d) Is it possible to find different (non-zero) starting populations M_1 and Y_1 so that the system is in equilibrium? That is, so that $M_1 = M_2 = M_3 = \dots$ and $Y_1 = Y_2 = Y_3 = \dots$? If so, give an example of such an M_1 and Y_1 .

Solution: Yes, you want to solve

$$\begin{bmatrix} \frac{9}{10} & 1 \\ \frac{1}{10} & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} M_1 \\ Y_1 \end{bmatrix}.$$

This gives equations

$$\begin{aligned}\frac{9}{10}M_1 + Y_1 &= M_1 \\ \frac{1}{10}M_1 &= Y_1.\end{aligned}$$

Solving these equations, we find that the solution set consists of all choices of M_1 and Y_1 satisfying $Y_1 = \frac{1}{10}M_1$. Any such choice will do, for example

$$\begin{aligned}M_1 &= 5000 \\ Y_1 &= 500.\end{aligned}$$