Math 304
Exam 2 Sample Problems Solutions
Final Version
March 28, 2004

1. Determine if the following sets of vectors are or are not vector spaces. If they are not, explain why.

   (a) $V =$ solution set of the equations $x + y - z - w = 0$ and $x + y + 2w = 0$ in $\mathbb{R}^4$.
   Solution: This is a vector space. Solution sets of homogeneous systems of linear equations are always vector spaces.

   (b) $W = \left\{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \,\middle|\, y = x + \frac{1}{x} \right\}$.
   Solution: W is not a vector space. Notice that $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] \in W$ but that $2 \cdot \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] \notin W$. So W is not closed under scalar multiplication.

   (c) $X =$ set of upper triangular 3×3 matrices.
   Solution: X is a vector space. We know that the set of 3×3 matrices forms a vector space. One needs only check that X is closed under addition and scalar multiplication. (On the exam, you would indeed want to check this for full credit!)

2. Determine if the following sets of vectors are or are not linearly independent. Explain how you know a set is or is not linearly independent. If a set is linearly dependent, then find a linear dependency among the vectors.

   (a) $\left\{ \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix} \right\}$
   Solution: These vectors are linearly independent. For two vectors to be linearly dependent, one must be a scalar multiple of the other. Any scalar multiple of $\left[\begin{smallmatrix} 1 \\ 0 \\ 9 \end{smallmatrix}\right]$ must have a 0 in the middle entry.

   (b) $\left\{ \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$
   Solution: These vectors are linearly independent. We check that

$$\det \begin{bmatrix} 3 & 1 & -1 \\ 4 & 0 & 2 \\ 5 & 9 & 0 \end{bmatrix} = -80 \neq 0.$$

   Thus these vectors are linearly independent by Thm. 3.3.1 in Leon.

   (c) $\left\{ \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution: These vectors must be linearly dependent. In a vector space of dimension 3, such as $\mathbb{R}^3$, any collection of 4 or more vectors must be linearly dependent (Thm. 3.4.1 in Leon). For a linear dependency we solve

$$a\begin{bmatrix}3\\4\\5\end{bmatrix}+b\begin{bmatrix}1\\0\\9\end{bmatrix}+c\begin{bmatrix}-1\\2\\0\end{bmatrix}+d\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

for a, b, c, d. This gives us three linear equations

$$\begin{aligned}3a+b-c+d&=0\\4a+2c+d&=0\\5a+9b+d&=0.\end{aligned}$$

Solving these equations we find $a=-5$, $b=1$, $c=2$, and $d=16$, is one of many possible solutions.

(d) $\left\{\begin{bmatrix}1\\3\\-4\\2\end{bmatrix},\begin{bmatrix}2\\2\\-4\\0\end{bmatrix},\begin{bmatrix}1\\-3\\2\\-4\end{bmatrix},\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\right\}$

Solution: As in part (b) we take the 4×4 determinant obtained from these vectors. However, here we find that the determinant is 0. This means that these vectors must be linearly dependent. Setting up the equations as we did in part (c), we find that one possible dependency is

$$4\begin{bmatrix}1\\3\\-4\\2\end{bmatrix}-3\begin{bmatrix}2\\2\\-4\\0\end{bmatrix}+2\begin{bmatrix}1\\-3\\2\\-4\end{bmatrix}+0\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

3. Find linear equations that define the subspaces that are the spans of the sets of vectors in (a)–(d) in problem 2.

(a) Solution: $\left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right]\in\mathrm{Span}(\left[\begin{smallmatrix}3\\4\\5\end{smallmatrix}\right],\left[\begin{smallmatrix}1\\0\\9\end{smallmatrix}\right])$ if and only if there are α, $\beta\in\mathbb{R}$ so that

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\alpha\begin{bmatrix}3\\4\\5\end{bmatrix}+\beta\begin{bmatrix}1\\0\\9\end{bmatrix}.$$

From this we get 3 equations,

$$\begin{aligned}x&=3\alpha+\beta\\y&=4\alpha\\z&=5\alpha+9\beta.\end{aligned}$$

The job is to eliminate α and β from these equations. What we do of course is form the augmented matrix

$$\left[\begin{array}{cc|c} 3 & 1 & x \\ 4 & 0 & y \\ 5 & 9 & z \end{array}\right].$$

By performing elementary row operations we obtain

$$\left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3}x \\ 0 & 1 & x - \frac{3}{4}y \\ 0 & 0 & 18x - 11y - 2z \end{array}\right].$$

The important row is the bottom row—in order for the equations to be consistent, the bottom row must be all 0's. Thus the Span$\left(\left[\begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ 9 \end{smallmatrix}\right]\right)$ is defined by

$$18x - 11y - 2z = 0.$$

(b) Solution: Since we have 3 linearly independent vectors in a 3-dimensional vector space ($\mathbb{R}^3$), they must span the whole space. Thus their span is $\mathbb{R}^3$.

(c) Solution: Since this set of vectors contains the vectors from (b) as a subset, its span must contain the span of the vectors from (b). Thus the span of these vectors contains all of $\mathbb{R}^3$, and so it must in fact be all of $\mathbb{R}^3$.

(d) We solve this problem just like we solved part (a). The augmented matrix we want obtain is

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & x \\ 3 & 2 & -3 & 0 & y \\ -4 & -4 & 2 & 1 & z \\ 2 & 0 & -4 & 0 & w \end{array}\right].$$

In row echelon form (almost) this is

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & x \\ 0 & -4 & -6 & 3 & y - 3x \\ 0 & 0 & 0 & -1 & w + x - y \\ 0 & 0 & 0 & 0 & x + y + z \end{array}\right].$$

Again in order to make the last row consistent, we must have $x+y+z = 0$, which is the equation for the span of these vectors.

4. Let

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Show that $\mathbf{v} = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right]$ is in the nullspace $N(A)$ of A.

Solution: Just do the multiplication!

(b) Find linear equations that define $N(A)$ as a subset of $\mathbb{R}^4$.

Solution: These can read off of the rows of A:

$$2x - y = 0$$
$$2y + z = 0.$$

(c) Find a basis of $N(A)$ that contains $\mathbf{v}$.

Solution: Our space $N(A)$ is the intersection of two hyperplanes in $\mathbb{R}^4$, and so it should be a 2-dimensional space. Thus its basis should have 2 elements in it. By solving the equations in part (b), we see that $N(A)$ is given by

$$N(A) = \left\{ \begin{bmatrix} -\frac{1}{4}\alpha \\ -\frac{1}{2}\alpha \\ \alpha \\ \beta \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}.$$

Let $\mathbf{u} = \begin{bmatrix} -1/4 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$.

Claim: $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $N(A)$. First, $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because neither one is a (non-zero) scalar multiple of the other. To check that $\text{Span}(\mathbf{u}, \mathbf{v}) = N(A)$ we observe that from the displayed line above,

$$\begin{aligned} N(A) &= \left\{ \alpha \begin{bmatrix} -1/4 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\} \\ &= \text{Span}(\mathbf{u}, \mathbf{v}). \end{aligned}$$

(d) What is the dimension of $N(A)$?

Solution: Since the basis of $N(A)$ has 2 vectors in it, the dimension of $N(A)$ is 2.

5. Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis of $\mathbb{R}^3$. Let $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

(a) Show that $\mathcal{A}$ is a basis for $\mathbb{R}^3$. Show that $\mathcal{B}$ is not a basis for $\mathbb{R}^3$.

Solution: To show that $\mathcal{A}$ is a basis for $\mathbb{R}^3$, it suffices to show that the vectors in $\mathcal{A}$ are linearly independent (we're making crucial use of Thm. 3.4.3 in Leon). To check that they are linearly independent, we take the determinant,

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2.$$

Since the determinant is $\neq 0$, the vectors in $\mathcal{A}$ are linearly independent.

To show that $\mathcal{B}$ is not a basis for $\mathbb{R}^3$, it suffices to show that the vectors in $\mathcal{B}$ are linearly dependent. Again, we take a determinant,

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 0.$$

Since the determinant is 0, the vectors made up from the columns are linearly dependent.

(b) Find a linear equation (or equations) whose set of solutions is Span($\mathcal{B}$).

Solution: We want to row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 1 & 0 & 1 & y \\ 0 & 1 & -1 & z \end{array}\right].$$

Performing elementary row operations we obtain,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & x-y \\ 0 & 0 & 0 & x-y-z \end{array}\right].$$

Thus,

$$\text{Span}(\mathcal{B}) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x - y - z = 0 \right\}.$$

(c) Find the coordinates of $\left[\begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}\right]$ with respect to the basis $\mathcal{A}$. (You may want to do the next part first.)

Solution: Let $\mathbf{v} = \left[\begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}\right]$. The change of basis matrix to convert from the $\mathcal{S}$-basis to the $\mathcal{A}$-basis is

$$U_{\mathcal{A}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since we need its inverse, we compute,

$$U_{\mathcal{A}}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

Thus we have that

$$
\begin{aligned}
[\mathbf{v}]_{\mathcal{A}} &= U_{\mathcal{A}}^{-1} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{A}} .
\end{aligned}
$$

(d) Find the change of basis matrix which converts coordinates in the $\mathcal{S}$-basis into coordinates in terms of the $\mathcal{A}$-basis.

Solution: This is the matrix $U_{\mathcal{A}}$ that was computed in part (c).