

1. [18 points]

(a) Find the following determinant. However, use only properties of determinants, without calculating directly (that is without expanding along a column or row or otherwise). Explain your answer.

$$\det \begin{bmatrix} 1 & 3 & -2 & -1 \\ 5 & 4 & 9 & -1 \\ 2 & 9 & 9 & 10 \\ 3 & -2 & 13 & 1 \end{bmatrix}$$

Solution: The determinant is 0, which can be verified in advance using your calculator. To show that it's 0 without expanding along columns or rows, you can do two things. First, if you row reduce the matrix into row echelon form, you obtain the matrix

$$\begin{bmatrix} 1 & 3 & -2 & -1 \\ 0 & 1 & 13/3 & 12/3 \\ 0 & 0 & 1 & 18/25 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix has determinant 0 by multiplying along the diagonal (since it's upper triangular). Two matrices that are row equivalent do not necessarily have the same determinant, but the determinant of one is a *non-zero* multiple of the determinant of the other. Thus, our original determinant must also be 0.

Or, you can solve the problem this way. Use your calculator to find that the determinant is 0 in advance. Once you know that the determinant should be 0, that means we should look for linear dependencies among the columns and among the rows. So then we look for one, and we find that

$$\text{Row 2} - 2 \cdot (\text{Row 1}) = \text{Row 4}.$$

Having a linear dependency among the rows then implies that the matrix has determinant 0, which is what we wanted to show.

(b) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 9 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 9 \\ 9 \\ 13 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 10 \\ 1 \end{bmatrix}.$$

Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ linearly independent or linearly dependent? Explain.

Solution: They must be linearly dependent. The matrix has determinant 0 (so is singular). The rows and columns of such a matrix must be linearly dependent.

(c) Using $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ from (b), is it the case that $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathbb{R}^4$?

Solution: No, they cannot span all of \mathbb{R}^4 . Any spanning set of \mathbb{R}^4 must contain at least 4 linearly independent vectors. Our set contains only 4 vectors, which are not linearly independent.

2. [14 points] Consider the following vectors in \mathbb{R}^3 :

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}.$$

(a) Are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_5$ linearly independent or linearly dependent? How do you know? If they are linearly dependent, find a non-trivial linear dependency among them.

Solution: They must be linearly dependent. The dimension of \mathbb{R}^3 is 3, so any set of 4 or more vectors must be linearly dependent. There are of course several dependencies to choose from, but here is one:

$$-\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 = \mathbf{0}.$$

(b) Find a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_5\}$ that is a basis of \mathbb{R}^3 . Justify your answer.

You need to find 3 of the vectors that are linearly independent. It turns out that any 3 of $\mathbf{v}_1, \dots, \mathbf{v}_5$ will do. We will pick $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. They will form a basis if we can show that they are linearly independent and that they span \mathbb{R}^3 . They are linearly independent because the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -3 & 0 \\ 2 & 5 & 5 \end{bmatrix}$$

has determinant -56 , which is $\neq 0$. Any three linearly independent vectors in \mathbb{R}^3 must also span \mathbb{R}^3 , so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must also span \mathbb{R}^3 .

3. [8 points] Consider the following subset of 2×2 matrices,

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid a + d = 0 \right\}.$$

(a) Is V a vector space? Explain.

Solution: Yes. It is closed under addition and scalar multiplication, and so it is a subspace of $\mathbb{R}^{2 \times 2}$.

(b) What is the dimension of V ?

Solution: The dimension is 3.

4. [14 points] Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & 5 & -1 \end{bmatrix}.$$

Find a basis for the nullspace $N(A)$. Justify your answer.

Solution: To find a basis for $N(A)$, we first put A in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 7 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The homogenous systems of linear equations defined by both matrices have the same set of solutions, so

$$N(A) = \left\{ \begin{bmatrix} -7\alpha + 2\beta \\ 2\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$

Therefore,

$$N(A) = \left\{ \alpha \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$

This gives us our basis: $\begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. These two vectors span $N(A)$ by the equation above. They are linearly independent because neither is a non-zero multiple of the other.

5. [20 points] Let $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$. Both \mathcal{A} and \mathcal{B} are bases for \mathbb{R}^3 .

(a) Why is \mathcal{A} a basis for \mathbb{R}^3 ?

Solution: The vectors in \mathcal{A} are linearly independent because the 3×3 matrix formed from them has determinant 1 (by multiplying down the diagonal). Any 3 linearly independent vectors in \mathbb{R}^3 must form a basis since \mathbb{R}^3 is 3-dimensional.

(b) What are the change of basis matrices $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ for \mathcal{A} and \mathcal{B} ?

Solution: The change of basis matrices are

$$U_{\mathcal{A}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad U_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Some people put the inverses of these matrices instead, which was fine for full credit on this part. Of course it was still important to use the right matrices in parts (c)–(d).

(c) Let $\mathbf{v} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. Express \mathbf{v} in terms of the basis \mathcal{A} . That is, find $[\mathbf{v}]_{\mathcal{A}}$.

Solution:

$$[\mathbf{v}]_{\mathcal{A}} = U_{\mathcal{A}}^{-1} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{A}}.$$

(d) Let $\mathbf{u} \in \mathbb{R}^3$ have representation $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}}$. What is the representation of \mathbf{u} in terms of the standard basis for \mathbb{R}^3 ?

Solution:

$$\mathbf{u} = U_{\mathcal{B}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix}.$$

(e) Using \mathbf{u} from part (d), what is $[\mathbf{u}]_{\mathcal{A}}$?

Solution:

$$[\mathbf{u}]_{\mathcal{A}} = U_{\mathcal{A}}^{-1} \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix} = \begin{bmatrix} 80 \\ -38 \\ 12 \end{bmatrix}.$$

6. [14 points] Determine whether each of the following statements is either **True** or **False**.

(a) Suppose S is a subset of \mathbb{R}^n so that $\text{Span}(S) = \mathbb{R}^n$. Then every vector $\mathbf{v} \in \mathbb{R}^n$ can be expressed as a linear combination of elements of S in only one way.

Solution: False. S must be a basis for this to be true.

(b) For vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$, the dimension of $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ is at least k .

Solution: False. The dimension is *at most* k .

(c) Suppose \mathbf{v}_1 and \mathbf{v}_2 are linearly independent vectors in \mathbb{R}^3 . Then there is a vector $\mathbf{v}_3 \in \mathbb{R}^3$ so that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

Solution: True. Any linearly independent set in a vector space can be enlarged to form a basis.

(d) Suppose S is a subset of a vector space V . If $\text{Span}(S) \neq V$, then there must be a vector $\mathbf{v} \in V$ such that $\mathbf{v} \notin \text{Span}(S)$.

Solution: True. If S is not a spanning set, then that means that there is definitely something in V that is not in the span of S .

(e) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4 \in \mathbb{R}^n$ are linearly dependent, then $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Solution: False. It would only be true if the non-trivial dependency among $\mathbf{v}_1, \dots, \mathbf{v}_4$ involves \mathbf{v}_4 with a non-zero coefficient.

(f) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and \mathbf{v}_4 be vectors in \mathbb{R}^n . Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly independent. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly independent.

Solution: False. It is easy to construct counterexamples in any vector space of dimension at least 3. For example, if n happens to be 3, then the four vectors must be linearly dependent.

(g) Suppose W is a subspace of a vector space V . If $\dim W = \dim V$, then $W = V$.

Solution: True. If both spaces have the same dimension, then there is a basis of W with the same number of elements as $\dim W = \dim V$. But any linearly independent subset of V containing the same number of elements as the dimension of V must also be a basis for V . So any basis for W is also a basis for V — i.e. W and V must be the same (they're both the spans of the same set).

7. [12 points] Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for a vector space V . Show that

$$\{\mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}, \mathbf{w}\}$$

is also a basis for V .

Solution: There are several ways to do this problem. Here is one way. First we show that $\mathbf{u} + \mathbf{v} + \mathbf{w}$, $\mathbf{v} + \mathbf{w}$, and \mathbf{w} are linearly independent. Suppose

$$a(\mathbf{u} + \mathbf{v} + \mathbf{w}) + b(\mathbf{v} + \mathbf{w}) + c\mathbf{w} = \mathbf{0},$$

with $a, b, c \in \mathbb{R}$. Then gathering terms:

$$a\mathbf{u} + (a + b)\mathbf{v} + (a + b + c)\mathbf{w} = \mathbf{0}.$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for V , we know that \mathbf{u} , \mathbf{v} , \mathbf{w} are linearly independent. Therefore the last equation implies that

$$a = 0$$

$$a + b = 0$$

$$a + b + c = 0.$$

We quickly find from this that the only solution is $a = b = c = 0$. Thus, $\mathbf{u} + \mathbf{v} + \mathbf{w}$, $\mathbf{v} + \mathbf{w}$, and \mathbf{w} are linearly independent, which is what we wanted to show.

To conclude that they form a basis, we need only observe that we have a set of 3 linearly independent vectors in a vector space of dimension 3. It must also span, so it is a basis.