Math 304
Exam 3 Study Sheet
April 13, 2004

Since the notation in the book is not entirely the same (or as useful) as what we have done in class, I've summarized the important things below. (At least for change of basis notation—this does not cover everything we did.)

Change of Basis Notation

Suppose $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$. We know that any vector $\mathbf{v} \in \mathbb{R}^n$ can be expressed *uniquely* in terms of $\mathcal{B}$:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1, \ldots, c_n \in \mathbb{R}$ are unique (i.e., once we choose $\mathbf{v}$, the c's are uniquely determined). We will write

$$\mathbf{v} = [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_{\mathcal{B}} \qquad (*)$$

to mean that $c_1, \ldots, c_n$ are the coordinates of $\mathbf{v}$ in terms of the basis $\mathcal{B}$.

Using the standard basis $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ for $\mathbb{R}^n$, then our usual coordinates are the stame as the coordinates with respect to $\mathcal{S}$. So

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{\mathcal{S}}.$$

Now if we let $U_{\mathcal{B}}$ be the $n \times n$ matrix formed by taking

$$U_{\mathcal{B}} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \vdots & \vdots & & \vdots \end{bmatrix}, \qquad (*)$$

where the columns of $U_{\mathcal{B}}$ are obtained by taking the column vectors for $\mathbf{v}_1, \ldots, \mathbf{v}_n$ under the standard basis and lining them up. The matrix $U_{\mathcal{B}}$ is called the *transition matrix* or *change of basis matrix.* Then for any vector $\mathbf{v} \in \mathbb{R}^n$ we find formulas,

$$[\mathbf{v}]_{\mathcal{S}} = U_{\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}, \qquad (*)$$

and likewise

$$[\mathbf{v}]_{\mathcal{B}} = U_{\mathcal{B}}^{-1} \cdot [\mathbf{v}]_{\mathcal{S}}, \qquad (*)$$

which allow us to change representations of $\mathbf{v}$ in terms of the standard basis and in terms of the basis $\mathcal{B}$ and back again.

If $\mathcal{A} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is another basis for $\mathbb{R}^n$, then we can interchange between $\mathcal{B}$-basis representations and $\mathcal{A}$-representations: for any $\mathbf{v} \in \mathbb{R}^n$,

$$[\mathbf{v}]_{\mathcal{A}} = U_{\mathcal{A}}^{-1} U_{\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$$

and likewise

$$[\mathbf{v}]_{\mathcal{B}} = U_{\mathcal{B}}^{-1} U_{\mathcal{A}} \cdot [\mathbf{v}]_{\mathcal{B}}.$$

Now suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Suppose that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a basis for $\mathbb{R}^m$. Then there is an $m \times n$ matrix $[L]_{\mathcal{B}}^{\mathcal{C}}$ so that

$$[L(\mathbf{x})]_{\mathcal{C}} = [L]_{\mathcal{B}}^{\mathcal{C}} [\mathbf{x}]_{\mathcal{B}}, \qquad \forall \mathbf{x} \in \mathbb{R}^n.$$

In fact,

$$[L]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ [L(\mathbf{v}_1)]_{\mathcal{C}} & [L(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{C}} \\ \vdots & \vdots & & \vdots \end{bmatrix}. \tag{*}$$

To think about what this means: $[L]_{\mathcal{B}}^{\mathcal{C}}$ takes as input vectors in coordinates with respect to $\mathcal{B}$, applies L to them, and then leaves as output vectors in coordinates with respect to $\mathcal{C}$.

Example 1: We know that any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is represented by an $m \times n$ matrix A:

$$L(\mathbf{x}) = A\mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^n,$$

and

$$A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) & \cdots & L(\mathbf{e}_n) \\ \vdots & \vdots & & \vdots \end{bmatrix}.$$

Everything here is done in terms of the standard basis. So what we have is

$$[L]_{\mathcal{S}_n}^{\mathcal{S}_m} = A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) & \cdots & L(\mathbf{e}_n) \\ \vdots & \vdots & & \vdots \end{bmatrix}. \tag{*}$$

Example 2: Let $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity transformation,

$$\mathcal{I}(\mathbf{x}) = \mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^n.$$

We can represent $\mathcal{I}$ in terms of a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, and we find

$$[\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}} = U_{\mathcal{B}} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \vdots & \vdots & & \vdots \end{bmatrix}, \tag{*}$$

where $\mathcal{S}$ is the standard basis on $\mathbb{R}^n$. Likewise,

$$[\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}} = ([\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}})^{-1} = U_{\mathcal{B}}^{-1}. \qquad (*)$$

If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, the representation of L in terms of the basis $\mathcal{B}$ is

$$\begin{aligned}[L]_{\mathcal{B}}^{\mathcal{B}} &= [\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}} \cdot [L]_{\mathcal{S}}^{\mathcal{S}} \cdot [\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}} \\ &= ([\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}})^{-1} \cdot [L]_{\mathcal{S}}^{\mathcal{S}} \cdot [\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}}. \qquad (*)\end{aligned}$$

For any linear transformation, $L : \mathbb{R}^n \to \mathbb{R}^m$, we can find $[L]_{\mathcal{S}_n}^{\mathcal{S}_m}$ from Example 1 above. If $\mathcal{B}$ is any basis of $\mathbb{R}^n$ and $\mathcal{C}$ is any basis of $\mathbb{R}^m$,

$$[L]_{\mathcal{B}}^{\mathcal{C}} = [\mathcal{I}]_{\mathcal{S}_m}^{\mathcal{C}} \cdot [L]_{\mathcal{S}_n}^{\mathcal{S}_m} \cdot [\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}_n}. \qquad (*)$$

Example 3: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$, and let $\mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \right\}$. Let $\mathcal{S}$ be the standard basis on $\mathbb{R}^3$. Things you should check:

$$[\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}} = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}; \qquad [\mathcal{I}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 3 & 9 \\ 2 & -2 & -3 \\ -1 & 2 & 4 \end{bmatrix}.$$

and

$$[L]_{\mathcal{S}}^{\mathcal{S}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 9 \end{bmatrix}; \qquad [L]_{\mathcal{S}}^{\mathcal{C}} = \begin{bmatrix} 3 & 7 & -3 \\ -1 & -1 & -3 \\ 1 & 2 & 3 \end{bmatrix}.$$

and

$$[L]_{\mathcal{B}}^{\mathcal{C}} = ?? \qquad [L]_{\mathcal{C}}^{\mathcal{B}} = ??$$

Additional Study Problems:

- 3.6: 1, 2, 7-11
- 4.1: 5, 7, 17
- 4.2: 1-10
- 4.3: 1, 2, 3, 4, 12