1. [28 points] Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 9 & 4 \\ -1 & -2 & -4 & 13 & 6 \\ 2 & 4 & 5 & 7 & 3 \end{bmatrix}.$$

(a) Compute the following four quantities:

- the dimension of the row space of A;
- the dimension of the column space of A;
- the dimension of the nullspace of A;
- the rank of A.

Explain your answers.

Solution: After putting A in row echelon form, we have the matrix

$$U = \begin{bmatrix} 1 & 2 & 2 & 9 & 4 \\ 0 & 0 & 1 & 11 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are 2 non-zero rows in this matrix, we know that

$$\text{dimension of the row space of } A = 2.$$

Then we know that the dimension of the row space of A is the same as the dimension of the column space of A as well as the rank of A. Therefore, both of these quantities are 2.

Finally the rank-nullity theorem says that

$$\text{rank}(A) + \dim(N(A)) = \#\text{columns of } A,$$

and so $\dim(N(A)) = 3$.

(b) Find bases for the following three spaces:

- the row space of A;
- the column space of A;
- the nullspace of A.

Explain your answers.

Solution: We know the dimensions of these spaces from part (a). At this point, several answers are possible for each space, but we'll give one solution here. The row space has as a basis the non-zero rows of U:

$$\{[1,2,2,9,4],[0,0,1,11,5]\}.$$

The columns space of A has as a basis the <u>columns from A</u> that contain the pivots from U:

$$\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}2\\-4\\5\end{bmatrix}\right\}.$$

Finally, we calculate the nullspace of A, by first putting U in reduced row echelon form:

$$U' = \begin{bmatrix}1 & 2 & 0 & 31 & 14\\ 0 & 0 & 1 & -11 & -5\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Then we find

$$N(A) = \left\{\begin{bmatrix}-2\alpha - 31\beta - 14\gamma\\ \alpha\\ 11\beta + 5\gamma\\ \beta\\ \gamma\end{bmatrix}\,\middle|\,\alpha,\beta,\gamma\in\mathbb{R}\right\}.$$

From this we see that the basis of the nullspace is the 3 linearly independent vectors

$$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-31\\0\\11\\1\\0\end{bmatrix},\begin{bmatrix}-14\\0\\5\\0\\1\end{bmatrix}\right\}.$$

2. [(a) & (b) 7 points each; (c) 5 points] Define a function $F:\mathbb{R}^3\to\mathbb{R}^2$ by

$$F\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_2+4x_3\\ x_1-3x_2-2x_3\end{bmatrix}.$$

(a) Show that F is a linear transformation.

Solution: We need to show that F respects addition and scalar multiplication. First,

$$\begin{aligned} F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) F\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) &= \begin{bmatrix} 2(x_2+y_2)+4(x_3+y_3) \\ (x_1+y_1)-3(x_2+y_2)-2(x_3+y_3) \end{bmatrix} \\ &= \begin{bmatrix} 2x_2+4x_3 \\ x_1-3x_2-2x_3 \end{bmatrix} + \begin{bmatrix} 2y_2+4y_3 \\ y_1-3y_2-2y_3 \end{bmatrix} \\ &= F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + F\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right). \end{aligned}$$

Likewise,

$$\begin{aligned} F\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) &= F\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} 2\alpha x_2+4\alpha x_3 \\ \alpha x_1-3\alpha x_2-2\alpha x_3 \end{bmatrix} \\ &= \alpha \begin{bmatrix} 2x_2+4x_3 \\ x_1-3x_2-2x_3 \end{bmatrix} \\ &= \alpha F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right). \end{aligned}$$

(b) Find a matrix A so that $F(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Solution: We have that

$$A = \Big[F(\mathbf{e}_1), F(\mathbf{e}_2), F(be_3)\Big] = \begin{bmatrix} 0 & 2 & 4 \\ 1 & -3 & -2 \end{bmatrix}.$$

(c) What are the dimensions of the kernel and the image of F? Find a basis for the kernel of F.

Solution: The kernel of F is the same as the nullspace of A. The image of F is the same as the column space of A. With this in mind, we row reduce A:

$$U = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

There are 2 pivots, so the dimension of the column space of A is 2 (so the dimension of the image is 2). The rank-nullity theorem says that

$$\operatorname{rank}(A) + \dim(N(A)) = 3.$$

Since the rank of A is 2, we have $\dim(N(A)) = 1$. We see that $\begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$ is in the nullspace of A, and since the space is 1-dimensional, this vector forms a basis.

3. [25 points] Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 3 \end{bmatrix}.$$

Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis of $\mathbb{R}^3$. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}$ be another basis of $\mathbb{R}^3$.

(a) What is the change of basis matrix $[\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}}$ that changes the coordinates of a vector from in terms of $\mathcal{B}$ to in terms of $\mathcal{S}$.

Solution: We line up the vectors in $\mathcal{B}$ into a 3×3 matrix:

$$[\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix}.$$

(b) What is $[\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}}$?

Solution: We know that $[\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}} = ([\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}})^{-1}$, so

$$[\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(c) Let $\mathbf{x}$ be the vector $\mathbf{x} = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix}_{\mathcal{S}}$ in terms of the standard basis. Find $[\mathbf{x}]_{\mathcal{B}}$, the coordinates for $\mathbf{x}$ in terms of $\mathcal{B}$.

Solution: We know that $[\mathbf{x}]_{\mathcal{B}} = [\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}}[\mathbf{x}]_{\mathcal{S}}$, so

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 \\ 17 \\ 5 \end{bmatrix}.$$

(d) Find a matrix that represents L in terms of the basis $\mathcal{B}$. That is, find $[L]_{\mathcal{B}}^{\mathcal{B}}$.

Solution: We know that $[L]_{\mathcal{B}}^{\mathcal{B}} = [\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}}[L]_{\mathcal{S}}^{\mathcal{S}}[\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}}$, and furthermore $[L]_{\mathcal{S}}^{\mathcal{S}} = A$. Therefore,

$$[L]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -4 & 2 \\ 20 & 12 & -5 \\ 14 & 7 & -2 \end{bmatrix}$$

(e) Using your computation of $[\mathbf{x}]_{\mathcal{B}}$ in (c) together with (d), find $[L(\mathbf{x})]_{\mathcal{B}}$.

Solution: We use that $[L(\mathbf{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}^{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$, so

$$[L(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} -6 & -4 & 2 \\ 20 & 12 & -5 \\ 14 & 7 & -2 \end{bmatrix} \begin{bmatrix} -9 \\ 17 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -17 \end{bmatrix}.$$

Multiple Choice: [4 points each] In each of Problems 4–7, circle the <u>best answer</u>.

4. Suppose $L : \mathbb{R}^4 \to \mathbb{R}^8$ is a linear transformation. The dimension of the image of L is at most

(A) 0

(B) 1

(C) 2

(D) 4

(E) 8

Solution: (D) is the best answer because the image can have no larger dimension than the domain $\mathbb{R}^4$. On the other hand, answering (E) yields a correct statement, and so partial credit of 2 points.

5. Suppose $L : \mathbb{R}^5 \to \mathbb{R}^3$ is a linear transformation. The dimension of the kernel of L is at least

(A) 0

(B) 1

(C) 2

(D) 3

(E) 5

Solution: (C) is the best solution: L is represented by a 3×5 matrix A. The column space of A has dimension at most 3, since L takes values in $\mathbb{R}^3$. Therefore the nullspace of A (= kernel of L) has dimension at least 2. Answering (A) or (B) does yield correct statements, and partial credit of 2 points.

6. Suppose that A is a 7×5 matrix. The maximum the rank of A can be is

(A) 0

(B) 1

(C) -2

(D) 5

(E) 7

Solution: The answer is (D) because the rank of A is the same as the dimension of the row space of A and the dimension of the column space of A. Since A has only 5 columns, its column space have dimension at most 5.

7. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, which is a basis for $\mathbb{R}^3$. Then the matrix $[\mathcal{I}]_{\mathcal{B}}^{\mathcal{B}}$ is

(A) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

(E) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: The answer is (E). We know that the matrix $[\mathcal{I}]_{\mathcal{B}}^{\mathcal{B}}$ is the matrix that transforms vectors from $\mathcal{B}$-coordinates to $\mathcal{B}$-coordinates. In other words, it does nothing, so it is the identity matrix. Another way of seeing this,

$$[\mathcal{I}]_{\mathcal{B}}^{\mathcal{B}} = ([\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}})^{-1}[\mathcal{I}]_{\mathcal{S}}^{\mathcal{S}}[\mathcal{I}]_{\mathcal{S}}^{\mathcal{B}}.$$

The middle matrix ($[\mathcal{I}]_{\mathcal{S}}^{\mathcal{S}}$) is just the identity matrix. Therefore $[\mathcal{I}]_{\mathcal{B}}^{\mathcal{B}} = ([\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}})^{-1}[\mathcal{I}]_{\mathcal{B}}^{\mathcal{S}}$, which itself is the identity matrix.

8. [12 points] Suppose we have a 3×1 matrix B and a 1×3 matrix C:

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}.$$

(a) Show that the 3×3 matrix BC has rank at most 1.

Solution: If all of the entries of B or all of the entries of C are 0, then BC is the zero matrix, and so has rank 0. So let's assume that B and C have at least on non-zero entry each. Then

$$BC = \begin{bmatrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & b_2c_2 & b_2c_3 \\ b_3c_1 & b_2c_3 & b_3c_3 \end{bmatrix}.$$

At least one column of BC is non-zero. The other two columns of BC are then multiples of the non-zero column. Thus the column span of BC has dimension 1, and so the rank of BC is 1.

(b) Suppose that A is a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

and suppose that A has rank 1. Show that there is a 3×1 matrix B and a 1×3 matrix C so that $A = BC$.

Solution: A has rank 1, so each of its rows are scalar multiples of a given row. For instance, we could have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ba_{11} & ba_{12} & ba_{13} \\ ca_{11} & ca_{12} & ca_{13} \end{bmatrix}$$

for some scalars b and c. From this we conclude that

$$A = \begin{bmatrix} 1 \\ b \\ c \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}.$$

The other possible cases are similar.