**Math 323**
**Exam 1 Sample Problems**
**Solution Guide**
September 30, 2013

Note that the following provides a guide to the solutions on the sample problems, but in some cases the complete solution would require more work or justification.

1. For the first system of equations:

   (a) $A = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$; and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$.

   (b) After performing the row reductions (which you need to write out), the reduced row echelon form is
   $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

   (c) The set of solutions is $\{(s, t, u) = (5, -9, 4)\}$.

   For the second system of equations:

   (a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ v \\ w \end{bmatrix}$; and $\mathbf{b} = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}$.

   (b) After performing the row reductions (which you need to write out), the reduced row echelon form is
   $$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & -6 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 10 \end{array}\right].$$

   (c) The set of solutions is $\{(-6 - \alpha + \beta, \alpha, 3 - \beta, 10 - \beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}$.

2. Take the determinants by expanding by minors along a row or column:

   (a) Determinant $= 0$; not invertible (since the determinant is 0).

   (b) Determinant $= 1$; invertible.

   (c) Determinant $= -1$; invertible.

3. Only the matrices in (b) and (c) are invertible. Find inverse by row reducing $[A \mid I]$ into reduced row echelon form, namely $[I \mid A^{-1}]$.

   (b) Inverse $= \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

(c) Inverse $= \begin{bmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{bmatrix}$

4. (a) Let $E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (b) Answers may vary. Can take $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$; then $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; then $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$; then $E_4 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; then $E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

5. Here we first calculate

$$AB = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

   Therefore, $(AB)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, and similarly $(AB)^3 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$. In general,

$$(AB)^n = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}.$$

6. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and so $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Thus,

$$\det(A) = \det(A^T) = ad - bc.$$

   Now since $AA^T = I$, we have that $A^T = A^{-1}$. Since

$$\det(A^{-1}) = \frac{1}{\det(A)},$$

   we see that

$$\det(A) = \det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)};$$

   that is $\det(A) = \frac{1}{\det(A)}$, which gives $\det(A)^2 = 1$. Thus, $\det(A) = \pm 1$.

7. The lower left entry of $A$ should be $-\cos(x)\sin(y)$; it has now been corrected in the problem set. Expand the first row by minors:

$$\begin{aligned} \det(A) &= \cos(y) \begin{vmatrix} \cos(x) & -\sin(x)\cos(y) \\ \sin(x) & \cos(x)\cos(y) \end{vmatrix} - 0 + \sin(y) \begin{vmatrix} \sin(x)\sin(y) & \cos(x) \\ -\cos(x)\sin(y) & \sin(x) \end{vmatrix} \\ &= \cos(y)(\cos^2(x)\cos(y) + \sin^2(x)\cos(y)) + \sin(y)(\sin^2(x)\sin(y) + \cos^2(x)\sin(y)) \\ &= \cos^2(y)(\cos^2(x) + \sin^2(x)) + \sin^2(y)(\sin^2(x) + \cos^2(x)) \\ &= \cos^2(y) \cdot 1 + \sin^2(y) \cdot 1 \\ &= 1. \end{aligned}$$

   So $\det(A) = 1$, regardless of $x$ and $y$.

8. Determine if the following sets of vectors are or are not vector spaces. If they are not, explain why.

   (a) $V =$ solution set of the equations $x + y - z - w = 0$ and $x + y + 2w = 0$ in $\mathbb{R}^4$.
   Solution: This is a vector space. Solution sets of homogeneous systems of linear equations are always vector spaces. (They are the nullspaces of the their corresponding coefficient matrices.)

   (b) $W = \left\{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mid y = x + \frac{1}{x} \right\}$.
   Solution: $W$ is not a vector space. Notice that $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \in W$ but that $2 \cdot \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right) \notin W$. So $W$ is not closed under scalar multiplication.

   (c) $X =$ set of upper triangular $3 \times 3$ matrices.
   Solution: $X$ is a vector space. We know that the set of $3 \times 3$ matrices forms a vector space. One needs only check that $X$ is closed under addition and scalar multiplication. (On the exam, you would indeed want to check this for full credit!)

9. (a) Check directly that $A\mathbf{x} = \mathbf{0}$.

   (b) The equations are $2x_1 - x_2 = 0$ and $2x_2 + x_3 = 0$.

   (c) We transform $A$ into reduce row echelon form to find that

$$A \longleftrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   In this case, $x_1$ and $x_2$ are leading variables, and $x_3$ and $x_4$ are free variables. S

$$N(A) = \left\{ \begin{bmatrix} -\frac{\alpha}{4} \\ -\frac{\alpha}{2} \\ \alpha \\ \beta \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}.$$

10. By symmetry, we need only show that if $A$ is row equivalent to $B$ then $B$ is row equivalent to $A$. Suppose that $A$ is row equivalent to $B$. Then there are elementary matrices $E_1$, $E_2, \ldots, E_k$, so that

$$(E_k \cdots E_1)A = B.$$

This implies that

$$(E_k \cdots E_1)^{-1}B = A.$$

Now

$$(E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1},$$

and the inverse of an elementary matrix is itself an elementary matrix. Therefore,

$$E_1^{-1} \cdots E_k^{-1} B = A,$$

and so $B$ is row equivalent to $A$.

11. By interchanging row 1 of $M$ with row $k+1$, row 2 with row $k+2$, and so on, we see that $M$ is row equivalent to $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Since we have made $k$ row swaps, we see that

$$\det(M) = (-1)^k \det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

(The following proof is good to know in principle, but in its entirety would be beyond the scope of an exam.) We now proceed by induction on $k$. If $k = 1$, then $A$ and $B$ are simply scalars ($1 \times 1$ matrices), and so $\det(M) = (-1)^k AB$ as desired. Suppose that the result is true for all $k$, $1 \leq k \leq \ell - 1$. Expand the determinant of $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ along the top row: letting $A_{ij}$ be the $ij$-minor of $A$, we see that

$$\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = a_{11} \det \begin{bmatrix} A_{11} & 0 \\ 0 & B \end{bmatrix} - a_{12} \det \begin{bmatrix} A_{12} & 0 \\ 0 & B \end{bmatrix} + \cdots + (-1)^{k+1} a_{1k} \det \begin{bmatrix} A_{1k} & 0 \\ 0 & B \end{bmatrix}.$$

Now expand each of the $B$'s along the bottom row. After the dust settles and you simplify the expressions, you obtain $\det \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right] = \det(A)\det(B)$.

12. Suppose that $S$ is a subspace of $\mathbb{R}^1$. Suppose that $S \neq \{0\}$. Therefore we can pick $x_0 \in S$ with $x_0 \neq 0$. Since $S$ is a subspace, it is closed under scalar multiplication, so for any $c \in \mathbb{R}$, we that $cx_0 \in S$. Now suppose $y \in \mathbb{R}$. We want to show that $y \in S$. Let $c = \frac{y}{x_0}$. Then

$$cx_0 = \frac{y}{x_0} \cdot x_0 = y \in S.$$

Therefore $\mathbb{R}^1 \subseteq S$, so $S = \mathbb{R}^1$.