Math 323
Final Exam Sample Problems
Answers and Suggestions
December 11, 2013

Note that the following provides answers to the sample problems and suggestions on their solutions. Complete solutions require more work/justification.

1. (a) The characteristic polynomial of A turns out to be $p_A(\lambda) = x - x^3$, and so the eigenvalues are $\boxed{\lambda = 0, 1, -1}$.

   (b) To find the eigenspace E_λ, we calculate the null space of $A - \lambda I$. The eigenspaces are $\boxed{E_{-1} = \text{Span}\{(0, -1, 1)^T\}}$, $\boxed{E_0 = \text{Span}\{(1, 0, 0)^T\}}$, $\boxed{E_1 = \text{Span}\{(1, -2, 1)^T\}}$.

   (c) We let Q be a matrix whose columns are linearly independent eigenvectors. So take

$$\boxed{\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix}}$$

   and we can check that $Q^{-1}AQ$ is a diagonal matrix whose entries are the eigenvalues of A.

   (d) Since $Q^{-1}AQ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ from part (c), we let D be the diagonal matrix on the right. Then

$$D^5 = (Q^{-1}AQ)^5 = (Q^{-1}AQ)(Q^{-1}AQ)(Q^{-1}AQ)(Q^{-1}AQ)(Q^{-1}AQ) = Q^{-1}A^5Q.$$

   From this we see that $\boxed{A^5 = QD^5Q^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}}$, which turns out to be the same as A.

2. (a) Suggestion: if $\lambda = 0$ is an eigenvalue, then $\det(A - \lambda I) = \det(A) = 0$.

   (b) Let $\mathbf{x}$ be an eigenvector with eigenvalue λ. So $A\mathbf{x} = \lambda\mathbf{x}$. Suggestion: if we multiply through by A^{-1}, we have $\mathbf{x} = \lambda A^{-1}\mathbf{x}$, and so $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$.

3. (a) After putting A in reduced row echelon form, we find that a basis of $N(A)$ is $\boxed{\{(5, 8, -14, 0)^T, (5, 22, 0, -14)^T\}}$.

   (b) By Theorem 5.2.1 in Leon, the orthogonal complement $N(A)^\perp$ is the same as the column space of the transpose of A. We find then that $\boxed{\{(4, 1, 2, 3)^T, (-2, 3, 1, 4)^T\}}$ is a basis.

4. We use properties of inner products and orthonormal bases, especially Theorem 5.5.2 and its corollaries from p. 243.

   (a) $\langle \mathbf{x}, \mathbf{y} \rangle = 6 - 2 - 4 = \boxed{0}$.

   (b) $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{4 + 4 + 1} = \boxed{3}$.

5. (a) Calculate the integral $\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(x)\cos(x)\,dx$, and show that it is 0.

   (b) We use here Theorem 5.5.2. In terms of the orthonormal set $\{\frac{1}{\sqrt{2}}, \cos(x), \cos(2x)\}$ we have

$$\sin^2(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 \cdot \cos(x) - \frac{1}{2} \cdot \cos(2x).$$

   Therefore, $\boxed{\langle \sin^2(x), \frac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}}}$, $\boxed{\langle \sin^2(x), \cos(x) \rangle = 0}$, $\boxed{\langle \sin^2(x), \cos(2x) \rangle = -\frac{1}{2}}$,

6. Use the Gram-Schmidt process, as outlined in the examples on pp. 261–263. The answers to this problem are in the back of the text.