**MATH 323.501**
**Exam 2 Solutions**
**November 12, 2013**

1. For each statement below, write down whether it is **true** or **false**.

(a) The rank of a matrix $B$ plus the dimension of the null space of $B$ is equal to the number of rows of $B$.

(b) If $L : V \to W$ is a linear transformation, then the kernel of $L$ is trivial if and only if $L$ is surjective.

(c) Suppose $V_1$ and $V_2$ are subspaces of $\mathbb{R}^4$ with $\dim(V_1) = 2$ and $\dim(V_2) = 3$. Then $\dim(V_1 \cap V_2)$ is either 1 or 2.

(d) Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly dependent vectors in $\mathbb{R}^n$. Then we have $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{Span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

(e) If $\lambda$ is an eigenvalue of a matrix $A$, then $A^2 - \lambda A$ is singular.

**Solutions:** (a) $\boxed{\text{False}}$: The rank plus the nullity is equal to the number of columns.

(b) $\boxed{\text{False}}$: The kernel is trivial if and only if $L$ is injective.

(c) $\boxed{\text{True}}$: Since $V_1 \cap V_2 \subset V_1$, and $\dim(V_1) = 2$, we know that $\dim(V_1 \cap V_2)$ is either 0, 1, or 2. We need to rule out that the dimension is 0. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of $V_1$ and let $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be a basis of $V_2$. There is a non-trivial linear dependcy among these 5 vectors. Use this linear dependency to show that there is a non-zero vector in $V_1 \cap V_2$.

(d) $\boxed{\text{False}}$: We know that at least one of $\mathbf{v}_1, \ldots, \mathbf{v}_4$ is redundant when taking the span, but we cannot say that it is definitly $\mathbf{v}_1$.

(e) $\boxed{\text{True}}$: Since $A^2 - \lambda A = A(A - \lambda I)$, it follows that $\det(A^2 - \lambda A) = \det(A)\det(A - \lambda I)$. Since $\lambda$ is an eigenvalue, we have $\det(A - \lambda I) = 0$, and so $\det(A^2 - \lambda A) = 0$.

2. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 - x_1 \\ 2 - x_2 \end{bmatrix}.$$

Is $L$ a linear transformation? Justify your answer.

**Solution:** It is $\boxed{\text{not a linear transformation}}$. The reason is that it fails to respect both addition and scalar multiplication. We only need to exhibit a counterexample to one of these, so we compute

$$L\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = L\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

$$3L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Since these two values are not the same, we see that $L$ does not respect scalar multiplication. Answers may vary of course, and it is also possible to provide a counterexample for addition.

3. Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 3 & 5 & 4 & 2 \\ 0 & 1 & 5 & 1 \end{pmatrix},$$

and let $V$ be the column space of $A$.

(a) Are the columns of $A$ linearly independent or linearly dependent? Justify your answer.

(b) Find a basis for $V$. Show your work.

(c) What is the rank of $A$? Explain.

**Solution:** (a) The columns are $\boxed{\text{linearly dependent}}$. The reason is that each column represents a vector in $\mathbb{R}^3$. Thus we are considering 4 vectors in this 3-dimensional space, so they must be linearly dependent.

(b) We perform row operations on $A$ and find that its reduced row echelon form is

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The leading 1's appear in the first two columns, so the first two columns of $A$ will make a basis for the column space of $A$. Thus $\boxed{\left\{ \left(\begin{smallmatrix} 1 \\ 3 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 5 \\ 1 \end{smallmatrix}\right) \right\}}$ is a basis.

(c) The rank of $A$ is $\boxed{2}$. We know that the rank of a matrix is the same as the dimension of its column space (or the dimension of its row space, or the number of non-zero rows when it is put in row echelon form).

4. Let $\mathcal{A} = \{ \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \} \subseteq \mathbb{R}^2$. We can take it as given that $\mathcal{A}$ is a basis for $\mathbb{R}^2$. Suppose that $\mathcal{B}$ is another basis of $\mathbb{R}^2$ with change of basis matrix

$$P = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.$$

That is, for every $\mathbf{v} \in \mathbb{R}^2$, we have $[\mathbf{v}]_{\mathcal{A}} = P\,[\mathbf{v}]_{\mathcal{B}}$.

(a) Find a matrix $Q$ such that for every $\mathbf{v} \in \mathbb{R}^2$, $[\mathbf{v}]_{\mathcal{B}} = Q\,[\mathbf{v}]_{\mathcal{A}}$.

(b) Suppose that $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and that with respect to the basis $\mathcal{A}$ we have $[L]_{\mathcal{A}}^{\mathcal{A}} = \left(\begin{smallmatrix} 1 & -2 \\ -3 & 4 \end{smallmatrix}\right)$. Find $[L]_{\mathcal{B}}^{\mathcal{B}}$.

(c) What are the vectors in $\mathcal{B}$ with respect to the standard coordinates on $\mathbb{R}^2$?

**Solution:** (a) If we take the equation $[\mathbf{v}]_{\mathcal{A}} = P[\mathbf{v}]_{\mathcal{B}}$ given and multiply through on the left by $P^{-1}$, we see that $P^{-1}[\mathbf{v}]_{\mathcal{A}} = [\mathbf{v}]_{\mathcal{B}}$. Thus $Q = P^{-1}$, which we calculate as $\boxed{Q = \left(\begin{smallmatrix} 3 & -5 \\ -1 & 2 \end{smallmatrix}\right)}$.

(b) The matrix $P$ is the change of basis matrix from basis $\mathcal{B}$ into basis $\mathcal{A}$. Thus $P = [I]^{\mathcal{A}}_{\mathcal{B}}$. Likewise, $Q = [I]^{\mathcal{B}}_{\mathcal{A}}$. Thus

$$[L]^{\mathcal{B}}_{\mathcal{B}} = [I]^{\mathcal{B}}_{\mathcal{A}} \, [L]^{\mathcal{A}}_{\mathcal{A}} \, [I]^{\mathcal{A}}_{\mathcal{B}} = Q \, [L]^{\mathcal{A}}_{\mathcal{A}} \, P.$$

We calculate and find

$$[L]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \boxed{\begin{pmatrix} 10 & 12 \\ -4 & -5 \end{pmatrix}}.$$

(c) If we let $V = [I]^{\mathcal{S}}_{\mathcal{A}}$ be the transition matrix from $\mathcal{A}$ to the standard basis and let $U = [I]^{\mathcal{S}}_{\mathcal{B}}$ be the transition matrix from $\mathcal{B}$ to the standard basis, then

$$P = [I]^{\mathcal{A}}_{\mathcal{B}} = [I]^{\mathcal{A}}_{\mathcal{S}} \, [I]^{\mathcal{S}}_{\mathcal{B}} = V^{-1}U.$$

Multiplying through by $V$ on the left we see that,

$$U = VP,$$

and the columns of $U$ will be the vectors in $\mathcal{B}$ with respect to the standard basis. Now $V = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, since those are the coordinates of the basis vectors in $\mathcal{A}$ with respect to the standard coordinates. Therefore,

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 1 & 2 \end{pmatrix},$$

and so with respect to the standard basis $\boxed{\mathcal{B} = \{ \left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 \\ 2 \end{smallmatrix}\right) \}}$.

5. Suppose $V$ is a vector space and that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $V$. Prove that $\{\mathbf{u} + 2\mathbf{v} + 3\mathbf{w}, 4\mathbf{v} + 5\mathbf{w}, 6\mathbf{w}\}$ is also a basis for $V$.

**Solution:** Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ contains 3 elements and is a basis for $V$, the dimension of $V$ is 3. Since $\dim(V) = 3$, any other set of 3 linearly independent vectors in $V$ will also be a basis. Thus it suffices to show that $\mathbf{u} + 2\mathbf{v} + 3\mathbf{w}$, $4\mathbf{v} + 5\mathbf{w}$, and $6\mathbf{w}$ are linearly independent.

Suppose that we have constants $a$, $b$, $c \in \mathbb{R}$ such that

$$a(\mathbf{u} + 2\mathbf{v} + 3\mathbf{w}) + b(4\mathbf{v} + 5\mathbf{w}) + c(6\mathbf{w}) = \mathbf{0}.$$

Then by manipulating the left-hand side and gathering terms we have

$$a\mathbf{u} + (2a + 4b)\mathbf{v} + (3a + 5b + 6c)\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent, this implies that

$$a = 0, \quad 2a + 4b = 0, \quad 3a + 5b + 6c = 0.$$

As $a = 0$, the second equation implies that $b = 0$ also. Then using the third equation, $a = b = 0$ implies that $c = 0$. Therefore, $\mathbf{u} + 2\mathbf{v} + 3\mathbf{w}$, $4\mathbf{v} + 5\mathbf{w}$, and $6\mathbf{w}$ are linearly independent.

6. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \\ -1 & 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of $A$.

(b) For each eigenvalue $\lambda$ of $A$, find vectors that span the eigenspace corresponding to $\lambda$.

**Solution:** (a) We calculate

$$\det \begin{pmatrix} 1-\lambda & 0 & -2 \\ 2 & -\lambda & -4 \\ -1 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)((-\lambda)(2-\lambda) - 2(-\lambda)) = -\lambda^2(\lambda - 3).$$

Therefore the eigenvalues are $\boxed{\lambda = 0, 3}$.

(b) $\underline{\lambda = 0}$: To calculate the eigenspace $E_0$ for $\lambda = 0$, we compute the null space of $A$ itself. The reduced row echelon form of $A$ is

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so the defining equation of the nullspace is $x_1 = 2x_3$. From this we see that $\boxed{E_0 = \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}}$.

$\underline{\lambda = 3}$: To calculate the eigenspace $E_3$ for $\lambda = 3$, we compute the null sapce of

$$A - 3I = \begin{pmatrix} -2 & 0 & -2 \\ 2 & -3 & -4 \\ -1 & 0 & -1 \end{pmatrix},$$

whose reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the defining equations of $E_3$ are $x_1 = -x_3$ and $x_2 = -2x_3$, so $\boxed{E_3 = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}}$.