Note that the following provides a guide to the solutions on the sample problems, but in some cases the complete solution would require more work or justification.

1. For the first system of equations:
   (a) \( A = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \); \( x = \begin{bmatrix} s \\ t \\ u \end{bmatrix} \); and \( b = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} \).

   (b) After performing the row reductions (which you need to write out), the reduced row echelon form is \( \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{bmatrix} \).

   (c) The set of solutions is \( \{(s, t, u) = (5, -9, 4)\} \).

For the second system of equations:
   (a) \( A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix} \); \( x = \begin{bmatrix} x \\ y \\ z \\ v \\ w \end{bmatrix} \); and \( b = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix} \).

   (b) After performing the row reductions (which you need to write out), the reduced row echelon form is \( \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & | & -6 \\ 0 & 0 & 1 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & 1 & 1 & | & 10 \end{bmatrix} \).

   (c) The set of solutions is \( \{(-6 - \alpha + \beta, \alpha, 3 - \beta, 10 - \beta, \beta) \mid \alpha, \beta \in \mathbb{R} \} \).

2. Take the determinants by expanding by minors along a row or column:
   (a) Determinant = 0; not invertible (since the determinant is 0).
   (b) Determinant = 1; invertible.
   (c) Determinant = -1; invertible.

3. Only the matrices in (b) and (c) are invertible. Find inverse by row reducing \([A \mid I]\) into reduced row echelon form, namely \([I \mid A^{-1}]\).
   (b) Inverse = \( \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \).
(c) Inverse =
\[
\begin{bmatrix}
-51 & 15 & 7 & 12 \\
31 & -9 & -4 & -7 \\
-10 & 3 & 1 & 2 \\
-3 & 1 & 1 & 1
\end{bmatrix}
\]

4. (a) Let 
\[
E = \begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
and 
\[
F = \begin{bmatrix}
1 & 0 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(b) Answers may vary. Can take 
\[
E_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 3 & 1
\end{bmatrix}
\]
; then 
\[
E_2 = \begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
; then 
\[
E_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & -1
\end{bmatrix}
\]

5. Here we first calculate
\[
AB = \begin{bmatrix}
-1 & 0 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
-2 & 0 \\
-1 & 1
\end{bmatrix}
= \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}
\]

Therefore, \((AB)^2 = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}\)^2 = \begin{bmatrix}
4 & 0 \\
0 & 4
\end{bmatrix}\), and similarly \((AB)^3 = \begin{bmatrix}
8 & 0 \\
0 & 8
\end{bmatrix}\). In general,
\[
(AB)^n = \begin{bmatrix}
2^n & 0 \\
0 & 2^n
\end{bmatrix}
\]

6. Suppose \(A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\), and so \(A^T = \begin{bmatrix}
a & c \\
b & d
\end{bmatrix}\). Thus,
\[
det(A) = det(A^T) = ad - bc.
\]

Now since \(AA^T = I\), we have that \(A^T = A^{-1}\). Since
\[
det(A^{-1}) = \frac{1}{det(A)};
\]
we see that
\[
det(A) = det(A^T) = det(A^{-1}) = \frac{1}{det(A)};
\]
that is \(det(A) = \frac{1}{det(A)}\), which gives \(det(A)^2 = 1\). Thus, \(det(A) = \pm 1\).

7. The lower left entry of \(A\) should be \(-\cos(x)\sin(y)\); it has now been corrected in the problem set. Expand the first row by minors:
\[
det(A) = \cos(y) \begin{vmatrix}
\cos(x) & -\sin(x) \\
\sin(x) & \cos(x)
\end{vmatrix} - \sin(y) \begin{vmatrix}
\sin(x) & \cos(x) \\
\cos(x) & \cos(y)
\end{vmatrix} - 0 + \sin(y) \begin{vmatrix}
\sin(x) & \sin(y) \\
\cos(x) & \cos(y)
\end{vmatrix} - \cos(x) \begin{vmatrix}
\sin(x) & \sin(y) \\
\cos(x) & \cos(y)
\end{vmatrix} \\
= \cos(y)(\cos^2(x)\cos(y) + \sin^2(x)\cos(y)) + \sin(y)(\sin^2(x)\sin(y) + \cos^2(x)\sin(y)) \\
= \cos^3(y)(\cos^2(x) + \sin^2(x)) + \sin^2(y)(\sin^2(x) + \cos^2(x)) \\
= \cos^2(y) \cdot 1 + \sin^2(y) \cdot 1 \\
= 1.
\]
So \(det(A) = 1\), regardless of \(x\) and \(y\).
8. Determine if the following sets of vectors are or are not vector spaces. If they are not, explain why.

(a) \( V = \) solution set of the equations \( x + y - z - w = 0 \) and \( x + y + 2w = 0 \) in \( \mathbb{R}^4 \).

Solution: This is a vector space. Solution sets of homogeneous systems of linear equations are always vector spaces. (They are the nullspaces of the their corresponding coefficient matrices.)

(b) \( W = \{ [x, y] \mid y = x + \frac{1}{2} \} \).

Solution: \( W \) is not a vector space. Notice that \( \left( \frac{1}{2} \right) \in W \) but that \( 2 \cdot \left( \frac{1}{2} \right) = \left( \frac{1}{1} \right) \notin W \). So \( W \) is not closed under scalar multiplication.

(c) \( X = \) set of upper triangular \( 3 \times 3 \) matrices.

Solution: \( X \) is a vector space. We know that the set of \( 3 \times 3 \) matrices forms a vector space. One needs only check that \( X \) is closed under addition and scalar multiplication. (On the exam, you would indeed want to check this for full credit!)

9. (a) Check directly that \( A\mathbf{x} = \mathbf{0} \).

(b) The equations are \( 2x_1 - x_2 = 0 \) and \( 2x_2 + x_3 = 0 \).

(c) We transform \( A \) into reduce row echelon form to find that

\[
A \leftrightarrow \begin{bmatrix}
1 & 0 & \frac{1}{4} & 0 \\
0 & 1 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

In this case, \( x_1 \) and \( x_2 \) are leading variables, and \( x_3 \) and \( x_4 \) are free variables. So

\[
N(A) = \left\{ \begin{bmatrix}
\alpha \\
\frac{-\alpha}{4} \\
\frac{\alpha}{2} \\
\beta
\end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.
\]

10. By symmetry, we need only show that if \( A \) is row equivalent to \( B \) then \( B \) is row equivalent to \( A \). Suppose that \( A \) is row equivalent to \( B \). Then there are elementary matrices \( E_1, E_2, \ldots, E_k \), so that

\[(E_k \cdots E_1)A = B.\]

This implies that

\[(E_k \cdots E_1)^{-1}B = A.\]

Now

\[(E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1},\]

and the inverse of an elementary matrix is itself an elementary matrix. Therefore,

\[E_1^{-1} \cdots E_k^{-1}B = A,\]

and so \( B \) is row equivalent to \( A \).
11. By interchanging row 1 of $M$ with row $k+1$, row 2 with row $k+2$, and so on, we see that $M$ is row equivalent to $[A\ 0 \ 0 \ B]$. Since we have made $k$ row swaps, we see that
\[ \det(M) = (-1)^k \det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}. \]

(The following proof is good to know in principle, but in its entirety would be beyond the scope of an exam.) We now proceed by induction on $k$. If $k = 1$, then $A$ and $B$ are simply scalars ($1 \times 1$ matrices), and so $\det(M) = (-1)^k AB$ as desired. Suppose that the result is true for all $k$, $1 \leq k \leq \ell - 1$. Expand the determinant of $[A\ 0 \ 0 \ B]$ along the top row: letting $A_{ij}$ be the $ij$-minor of $A$, we see that
\[ \det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = a_{11} \det \begin{bmatrix} A_{11} & 0 \\ 0 & B \end{bmatrix} - a_{12} \det \begin{bmatrix} A_{12} & 0 \\ 0 & B \end{bmatrix} + \cdots + (-1)^{k+1}a_{1k} \det \begin{bmatrix} A_{1k} & 0 \\ 0 & B \end{bmatrix}. \]

Now expand each of the $B$’s along the bottom row. After the dust settles and you simplify the expressions, you obtain $\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A) \det(B)$.

12. Suppose that $S$ is a subspace of $\mathbb{R}^1$. Suppose that $S \neq \{0\}$. Therefore we can pick $x_0 \in S$ with $x_0 \neq 0$. Since $S$ is a subspace, it is closed under scalar multiplication, so for any $c \in \mathbb{R}$, we have $cx_0 \in S$. Now suppose $y \in \mathbb{R}$. We want to show that $y \in S$. Let $c = \frac{y}{x_0}$. Then
\[ cx_0 = \frac{y}{x_0} \cdot x_0 = y \in S. \]

Therefore $\mathbb{R}^1 \subseteq S$, so $S = \mathbb{R}^1$. 

\[ \text{4} \]