This exam consists of 5 problems, numbered 1–5. For partial credit you must present your work clearly and understandably and justify your answers.

The use of calculators is not permitted on this exam.

The point value for each question is shown next to each question.

<table>
<thead>
<tr>
<th>Points Possible</th>
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<tr>
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<td>Total</td>
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CHECK THIS EXAMINATION BOOKLET BEFORE YOU START. THERE SHOULD BE 5 PROBLEMS ON 5 PAGES (INCLUDING THIS ONE).

Do not mark in the box below.
1. [15 points] For each statement below, write down whether it is **true** or **false**.

(a) The function $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (x-y)^2$ is a linear transformation.

(b) For a $2 \times 3$ matrix $A$, it is possible for the column space to have dimension 3.

(c) For a $2 \times 3$ matrix $A$, the dimension of $N(A)$ can be 1, 2 or 3, but not 0.

(d) For a $3 \times 5$ matrix $C$, if for every $b \in \mathbb{R}^3$ the system $Cx = b$ is consistent, then the rank of $C$ is 3.

(e) Suppose that $v_1, v_2, v_3$ are linearly independent vectors in $\mathbb{R}^6$. Suppose that $w_1, w_2, w_3$ are linearly independent vectors in $\mathbb{R}^6$. Then together $v_1, v_2, v_3, w_1, w_2, w_3$ are linearly independent vectors in $\mathbb{R}^6$.

2. [8 points] Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation defined by

$$L \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$ 

Find a matrix $A$ so that $L(x) = Ax$ for all $x \in \mathbb{R}^2$. 

April 8, 2015
3. [20 points; (a) & (b) 6 pts. each; (c) & (d) 4 pts. each] Let \( A \) be the matrix
\[
A = \begin{pmatrix}
1 & 0 & 1 \\
3 & 6 & 0 \\
-1 & -6 & 2
\end{pmatrix},
\]
let \( V \) be the column space of \( A \), and let \( W \) be the row space of \( A \).

(a) Find a basis for \( V \). Show your work.

(b) Find a basis for \( W \). Show your work.

(c) What is the rank of \( A \)? Explain.

(d) What is the dimension of the null space of \( A \)? Explain.
4. [18 points] As usual we let \( S = \{e_1, e_2\} \) denote the standard basis for \( \mathbb{R}^2 \). Suppose that \( B = \{u_1, u_2\} \) is another basis for \( \mathbb{R}^2 \), and suppose we have a change of basis matrix

\[
P = [I]^S_B = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}.
\]

That is, for every \( x \in \mathbb{R}^2 \), we have \([x]_B = P[x]_S\).

(a) Find a matrix \( Q \) such that for every \( x \in \mathbb{R}^2 \), \([x]_S = Q[x]_B\).

(b) What are \( u_1 \) and \( u_2 \) with respect to the standard coordinates on \( \mathbb{R}^2 \)?

(c) Suppose that \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear transformation and that with respect to the basis \( S \) we have \([L]_S^S = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}\). Find \([L]_B^S\).
5. [12 points] Let $S_1$ and $S_2$ be finite subsets of a vector space $V$ such that $S_1 \subseteq S_2$. Let $d_1$ be the dimension of $\text{Span}(S_1)$, and let $d_2$ be the dimension of $\text{Span}(S_2)$.

(a) Prove that $d_1 \leq d_2$.

(b) Prove that if $\text{Span}(S_1) = V$, then $d_1 = d_2$. 