Note that the following provides answers to the sample problems and suggestions on their solutions. Complete solutions may require more work/justification.

1. It is not a linear transformation. The reason is that it fails to respect both addition and scalar multiplication. We only need to exhibit a counterexample to one of these, so we compute

\[
L \left( \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right) = L \left( \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]

and

\[
3L \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.
\]

Since these two values are not the same, we see that \( L \) does not respect scalar multiplication.

Answers may vary of course, and it is also possible to provide a counterexample for addition.

2. (a) The characteristic polynomial of \( A \) turns out to be \( p_A(\lambda) = x - x^3 \), and so the eigenvalues are \( \lambda = 0, 1, -1 \).

(b) To find the eigenspace \( E_\lambda \), we calculate the solution set of \((A - \lambda I)x = 0\). The eigenspaces are \( E_{-1} = \text{Span}\{(0, -1, 1)^T\}, \ E_0 = \text{Span}\{(1, 0, 0)^T\}, \ E_1 = \text{Span}\{(1, -2, 1)^T\} \).

(c) We let \( Q \) be a matrix whose columns are linearly independent eigenvectors. So take

\[
\begin{pmatrix}
0 & 1 & 1 \\
-1 & 0 & -2 \\
1 & 0 & 1
\end{pmatrix}
\]

and we can check that \( Q^{-1}AQ \) is a diagonal matrix whose entries are the eigenvalues of \( A \).

(d) Since \( Q^{-1}AQ = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \) from part (c), we let \( D \) be the diagonal matrix on the right. Then

\[
D^5 = (Q^{-1}AQ)^5 = (Q^{-1}AQ)(Q^{-1}AQ)(Q^{-1}AQ)(Q^{-1}AQ)(Q^{-1}AQ) = Q^{-1}A^5Q.
\]

From this we see that \( A^5 = QD^5Q^{-1} = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 2 \\
0 & -1 & -2
\end{pmatrix} \), which turns out to be the same as \( A \).

3. There are a couple approaches here. The easiest to see what is going on is to multiply out the right-hand side: you find

\[
L \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 6x_1 + 2x_2 \end{bmatrix}.
\]
From this we see that

\[ A = \begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix}. \]

4. (a) We see that \( Q = P^{-1}, \) so

\[ Q = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}. \]

(b) The standard coordinates of \( u_1 \) and \( u_2 \) are then the columns of \( Q:\)

\[ u_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -7 \\ 4 \end{bmatrix}. \]

(c) Suppose \( C \) is the matrix we are looking for. Then \( C = PAQ, \) so

\[ C = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -7 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}. \]

5. (a) Suppose that \( \lambda = 0 \) is an eigenvalue of \( L. \) Then there is a non-zero eigenvector \( v \in V \) with eigenvalue 0. Thus

\[ L(v) = 0 \cdot v = 0, \]

and therefore \( v \in \ker(L). \) Thus \( \ker(L) \neq \{0\}, \) and so \( L \) is not one-to-one. Since \( L \) is invertible however, it must be one-to-one, and so we have a contradiction.

(b) Let \( v \) be an eigenvector with eigenvalue \( \lambda. \) Thus \( L(v) = \lambda v. \) Let \( w = \lambda v = L(v). \) Then

\[ L^{-1}(w) = v = \frac{1}{\lambda} \cdot w. \]

Thus \( w \) is an eigenvector for \( L^{-1} \) with eigenvalue \( \frac{1}{\lambda}. \)

6. It may help if we write things in terms of column vectors, and we notice that for

\[ A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}, \]

we have

\[ L([x, y, z]) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - z \\ x + y \\ y + z \\ x - y - 2z \end{bmatrix}. \]

We find that the reduced row echelon form of \( A \) is the matrix

\[ B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]
Thus the kernel of $L$ consists of vector $[x, y, z]$ with $x - z = 0$ and $y + z = 0$, so
\[
\ker(L) = \{[z, -z, z] \mid z \in \mathbb{R}\} = \{z[1, -1, 1] \mid z \in \mathbb{R}\}.
\]
From this we see that $\{[1, -1, 1]\}$ is a basis for $\ker(L)$. The range of $L$ is given by the column space of $A$. We know that a basis of the column space can be found by taking the columns in $A$ corresponding to the pivots in $B$. Thus $\{[1, 1, 0, 1], [0, 1, 1, -1]\}$ is a basis for $\text{range}(L)$.

7. We assume that $m > n$. (a) Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. By the Dimension Theorem
\[
\dim(ker(L)) + \dim(range(L)) = n,
\]
and so $\dim(range(L)) \leq n < m$. Thus the range of $L$ cannot be all of $\mathbb{R}^m$.
(b) Suppose $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. By the Dimension Theorem
\[
\dim(ker(L)) + \dim(range(L)) = m.
\]
Now $\dim(range(L)) \leq n$, and so
\[
\dim(ker(L)) = m - \dim(range(L)) \geq m - n > 0.
\]
Therefore the kernel of $L$ is non-trivial, and so $L$ cannot be one-to-one.

8. Suppose that $v \in \ker(L)$ with $v \neq 0$. Now let $T = \{v\}$. Since $v \neq 0$, $T$ is a linearly independent set. By hypothesis, $L(T) = \{L(v)\}$ is a linearly independent set, but $L(v) = 0$, so $L(T) = \{0\}$. Since $\{0\}$ is not a linearly independent set, we have a contradiction. Therefore $\ker(L) = \{0\}$ and so $L$ is one-to-one.

9. I leave it to you to show that $L$ is a linear operator. Suppose that $p \in P_n$, so $p$ is a polynomial of degree $d$, with $d < n$. Now
\[
L(p) = p + p',
\]
and $p + p'$ also has degree $d$, since $p'$ has degree strictly smaller that $p$. Thus if $p \neq 0$, we see that $p + p'$ is also non-zero. Thus $L(p) = 0$ if and only if $p = 0$, and so $L$ is one-to-one (its kernel is trivial). To show that $L$ is onto, we want to show that for any polynomial $q \in P_n$, there is another polynomial $p \in P_n$ with
\[
L(p) = p + p' = q.
\]
Suggestion: Show this by induction on the degree of $q$. 