MATH 323.503
Exam 2 Solutions
November 18, 2015

1. Consider the matrix

\[ A = \begin{bmatrix} 8 & -6 \\ 9 & -7 \end{bmatrix}. \]

(a) Determine the eigenvalues of \( A \).

(b) For each eigenvalue of \( A \) find the corresponding fundamental eigenvectors.

(c) Find a matrix \( P \) so that \( P^{-1}AP = D \) is a diagonal matrix. What is \( D \)?

Solution: (a) To find the eigenvalues we determine the characteristic polynomial of \( A \):

\[ p_A(x) = |A - xI| = \begin{vmatrix} 8 - x & -6 \\ 9 & -7 - x \end{vmatrix} = -(8 - x)(7 + x) + 54 = x^2 - x - 2. \]

This then factors as

\[ p_A(x) = x^2 - x - 2 = (x - 2)(x + 1), \]

and so the eigenvalues of \( A \) are \( \lambda = -1, 2 \).

(b) For \( \lambda = -1 \), we take the homogeneous system of equations whose augmented matrix is

\[ \begin{bmatrix} 8 - (-1) & -6 \\ 9 & -7 - (-1) \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 9 & -6 \end{bmatrix}. \]

The reduced row echelon form is then quickly found to be

\[ \begin{bmatrix} 1 & \frac{-2}{3} \\ 0 & 0 \end{bmatrix}. \]

Thus \[ \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \] is an eigenvector with eigenvalue \( \lambda = -1 \). Optionally we clear the denominator to take \( v_1 = \begin{bmatrix} \frac{2}{3} \end{bmatrix} \) for our eigenvector.

For \( \lambda = 2 \), we similarly take

\[ \begin{bmatrix} 8 - 2 & -6 \\ 9 & -7 - 2 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 9 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}. \]

Thus \( v_2 = \begin{bmatrix} 1 \end{bmatrix} \) is an eigenvector with eigenvalue \( \lambda = 2 \).

(c) We know that if we take \( P \) to be a square matrix whose columns are linearly independent eigenvectors for \( A \), then \( P^{-1}AP \) will be diagonal. So we take

\[ P = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}. \]
Furthermore we know that $P^{-1}AP$ will be a diagonal matrix with the eigenvalues of $A$ on the diagonal, in the same order as the eigenvectors of $A$ appear in $P$. That is,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Note that no computation is required, since we know from the theory that these matrices will work. However, one can easily check that $P^{-1}AP = D$ to verify that we have the right answer.

2. For each statement below, write down whether it is True or False.

(a) If $\lambda$ is an eigenvalue of a matrix $A$, then $\lambda + 1$ is an eigenvalue of $A + I$ (where $I$ is the identity matrix).

(b) Let $S = \{v_1, v_2, v_3\}$ be a subset of a 3-dimensional vector space $V$. If $v_1 \notin \text{Span}(v_2)$ and $v_3 \notin \text{Span}(v_1, v_2)$, then $S$ spans $V$.

(c) Suppose that $S$ and $T$ are linearly independent subsets of a vector space $V$. Then $S \cup T$ is a linearly independent set.

(d) Let $T = \{w_1, w_2, w_3, w_4\}$ be a subset of a vector space $W$. If every vector in $W$ can be expressed as a linear combination of the elements of $T$, then the dimension of $W$ is 4.

**Solution:** Credit was given for the correct answers. The explanations were not necessary, but the ones below may answer some of your questions about these problems.

(a) **True**: If $\lambda$ is an eigenvalue of $A$, then $A$ has an eigenvector $x$ so that $Ax = \lambda x$. Then we note that $(A + I)x = Ax +Ix = \lambda x + x = (\lambda + 1)x$. Thus $\lambda + 1$ is an eigenvalue of $A + I$.

(b) **True**: The conditions in the problem imply that $S = \{v_1, v_2, v_3\}$ is a linearly independent set. Since dim($V$) = 3, any set of 3 elements must be a basis, and thus $S$ must also span $V$.

(c) **False**: It suffices to exhibit a counterexample. Let $V = \mathbb{R}^2$ and take $S = \{[1, 0], [0, 1]\}$ and take $T = \{[1, 1]\}$. So $S \cup T = \{[1, 0], [0, 1], [1, 1]\}$, which consists of 3 vectors in $\mathbb{R}^2$, and so must be linearly dependent since dim($\mathbb{R}^2$) = 2. (Moreover, $[1, 0] + [0, 1] + (-1)[1, 1] = [0, 0]$ provides the linear dependency.)

(d) **False**: The dimension of $W$ will be 4 only if $T$ is also linearly independent in addition to spanning $W$. However, it may not be, so we cannot conclude that dim($W$) = 4.

3. Consider the subset $S = \{[1, 1, 0], [21, 26, 0], [13, -7, 0], [e, \pi, 0]\} \subseteq \mathbb{R}^3$. Answer the following questions without doing any lengthy calculations or matrix row reductions.

(a) Does $S$ span $\mathbb{R}^3$? Why or why not?

(b) Is $S$ linearly independent? Why or why not?
Solution: (a) \[ \text{No}. \] Span(S) consists of all linear combinations of the vectors in S, but the last coordinate of any linear combination of the elements of S will necessarily be 0. Thus not all vectors in \( \mathbb{R}^3 \) are in the span of S.

(b) \[ \text{No}. \] We know that \( \text{dim}(\mathbb{R}^3) = 3 \), and so any set of 4 or more distinct vectors in \( \mathbb{R}^3 \) must be linearly dependent.

4. Consider the matrix
\[
B = \begin{bmatrix}
1 & -3 & 0 & 9 \\
0 & 0 & 1 & -2 \\
0 & 0 & 2 & -4
\end{bmatrix}.
\]

(a) Let \( R \) be the row space of \( B \). Find a basis of \( R \). Justify your answer.

(b) Let \( V = \{ \mathbf{x} \in \mathbb{R}^4 \mid B\mathbf{x} = \mathbf{0} \} \). Find a basis of \( V \). Justify your answer.

Solution: We first see that the reduced row echelon form of \( B \) is the matrix
\[
R = \begin{bmatrix}
1 & -3 & 0 & 9 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

(a) We know that the non-zero rows of \( R \) will be a basis of the row space of \( B \), so \( \{ [1, -3, 0, 9], [0, 0, 1, -2] \} \) is a basis of the row space.

(b) From \( R \) we see that \( V \) consists of vectors \( \begin{bmatrix} x_1 \\
  x_2 \\
 x_3 \\
 x_4 \end{bmatrix} \) satisfying
\[
x_1 - 3x_2 + 9x_4 = 0 \\
x_3 - 2x_4 = 0.
\]

The independent variables are \( x_2 \) and \( x_4 \), and so we see that
\[
V = \left\{ \begin{bmatrix} 3a - 9b \\
 a \\
 0 \\
 b \end{bmatrix} \mid a, b \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 3 \\
 1 \\
 0 \\
 0 \end{bmatrix} + b \begin{bmatrix} -9 \\
 0 \\
 2 \\
 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.
\]

The resulting fundamental solutions of \( A\mathbf{x} = \mathbf{0} \) form a basis of \( V \), and so
\[
\left\{ \begin{bmatrix} 3 \\
 1 \\
 0 \\
 0 \end{bmatrix}, \begin{bmatrix} -9 \\
 0 \\
 2 \\
 1 \end{bmatrix} \right\}
\]
is a basis.
5. Let $B = (u_1, u_2) = ([5], [1])$ and $C = (v_1, v_2) = ([2], [1])$ be ordered bases of $\mathbb{R}^2$. We can take it as given that both $B$ and $C$ are bases for $\mathbb{R}^2$. Let $S = (e_1, e_2)$ be the standard basis of $\mathbb{R}^2$.

(a) Find the transition matrix $P$ from the basis $B$ to the standard basis $S$. That is, find a matrix $P$ so that for all $x \in \mathbb{R}^2$, we have $P[x]_B = [x]_C$.

(b) Find the transition matrix $Q$ from the standard basis $S$ to the basis $C$.

(c) Use your findings in (a) and (b) to find the transition matrix from the basis $B$ to the basis $C$.

(d) Let $x = 2u_1 - 3u_2$. What is $[x]_C$?

**Solution:**

(a) We know that $P$ is formed by lining up the vectors from $B$:

$$P = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}.$$ 

(b) The problem here for $C$ is similar to the one in (a), but in reverse order. Thus, we also need to take an inverse,

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$ 

(c) The matrix $P$ transitions from $B$-coordinates to $S$-coordinates, and $Q$ transitions from $S$-coordinates to $C$-coordinates, so $QP$ will transition from $B$-coordinates to $C$-coordinates:

$$QP = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -5 & 1 \end{bmatrix}.$$ 

(d) Since $x = 2u_1 - 3u_2$, we see that with respect to $B$ we have coordinates

$$[x]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$ 

Changing to $C$-coordinates (using (c)),

$$[x]_C = QP[x]_B = \begin{bmatrix} 5 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ -13 \end{bmatrix}.$$ 

It is not necessary for the solution, but we can check that we are correct by observing that $x = 2[5] - 3[1] = [7]$ and that $10v_1 - 13v_2 = 10[5] - 13[1] = [7]$. 


6. Let \( W_1 \) and \( W_2 \) be subspaces of \( \mathbb{R}^6 \), with \( \dim(W_1) = 2 \) and \( \dim(W_2) = 4 \).

(a) Prove that \( W_1 \cap W_2 \) is a subspace of \( \mathbb{R}^6 \).

(b) Let \( B_1 \) and \( B_2 \) be bases for \( W_1 \) and \( W_2 \) respectively. Suppose that \( \text{Span}(B_1 \cup B_2) = \mathbb{R}^6 \). Prove that \( W_1 \cap W_2 = \{0\} \).

**Solution:**

(a) We need to prove that \( W_1 \cap W_2 \) is nonempty, closed under addition, and closed under scalar multiplication. Since \( W_1 \) and \( W_2 \) are both subspaces of \( \mathbb{R}^6 \), each must contain the zero vector \( 0 \). Thus \( 0 \in W_1 \cap W_2 \), and so \( W_1 \cap W_2 \) is nonempty.

To see that \( W_1 \cap W_2 \) is closed under addition, we let \( x, y \in W_1 \cap W_2 \) be arbitrary. We note that \( x + y \in W_1 \), since \( W_1 \) is a subspace, and likewise \( x + y \in W_2 \), since \( W_2 \) is a subspace. Therefore, \( x + y \in W_1 \cap W_2 \), and so \( W_1 \cap W_2 \) is closed under addition.

To see that \( W_1 \cap W_2 \) is closed under scalar multiplication, we let \( x \in W_1 \cap W_2 \) and \( c \in \mathbb{R} \) be arbitrary. Now \( cx \in W_1 \), since \( W_1 \) is a subspace, and likewise \( cx \in W_2 \), since \( W_2 \) is a subspace. Therefore \( cx \in W_1 \cap W_2 \), and so \( W_1 \cap W_2 \) is closed under scalar multiplication.

(b) We first observe that \( |B_1| = 2 \), since \( B_1 \) is a basis of \( W_1 \) and \( \dim(W_1) = 2 \), and that \( |B_2| = 4 \), since \( B_2 \) is a basis of \( W_2 \) and \( \dim(W_2) = 4 \). Thus \( |B_1 \cup B_2| \) is at most \( 6 \). However, we are given that \( \text{Span}(B_1 \cup B_2) = \mathbb{R}^6 \), and since the size of a spanning set must be no smaller than the dimension of the vector space it spans, it must be that

\[
|B_1 \cup B_2| \geq 6.
\]

Thus, \( |B_1 \cup B_2| = 6 \) and \( B_1 \cup B_2 \) spans \( \mathbb{R}^6 \), which has dimension 6. Therefore, \( B_1 \cup B_2 \) is a basis of \( \mathbb{R}^6 \).

Now suppose that \( x \in W_1 \cap W_2 \). We want to show that \( x = 0 \). Suppose that \( B_1 = \{u_1, u_2\} \) and \( B_2 = \{v_1, v_2, v_3, v_4\} \). Since \( x \in W_1 \) and \( W_1 = \text{Span}(B_1) \), we see that

\[
x = a_1 u_1 + a_2 u_2,
\]

for some \( a_1, a_2 \in \mathbb{R} \). Likewise, since \( x \in W_2 \), we have

\[
x = b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4,
\]

for some \( b_1, b_2, b_3, b_4 \in \mathbb{R} \). Subtracting we find

\[
a_1 u_1 + a_2 u_2 - b_1 v_1 - b_2 v_2 - b_3 v_3 - b_4 v_4 = 0,
\]

but since \( B_1 \cup B_2 = \{u_1, u_2, v_1, v_2, v_3, v_4\} \) is a basis of \( \mathbb{R}^6 \), it is a linearly independent set. Therefore, it must be that \( a_1 = a_2 = b_1 = b_2 = b_3 = b_4 = 0 \). In particular \( x = 0 \).