

Math 415
HW #6 Solutions, Version 2
October 19, 2004

In the problems that are to be turned in for a grade (the ‘*’ problems), I give a hint. For the other problems I explain how to solve the problem, though a proper write-up will need more details and explanation.

Section 6, #4. Since $50/8 = 6.25$ we see that the quotient is 6, and the remainder is 4: i.e. $50 = 8 \cdot 6 + 2$.

18. Hint: Use Theorem 6.14 in the book.

20. Hint: This will be a finite cyclic group consisting of roots of unity. Which ones?

23. By Theorem 6.14, we expect that \mathbb{Z}_{36} will have a unique subgroup for each positive divisor of 36. That is, looking at the subgroups generated by 1, 2, 3, 4, 6, 9, 12, 18, and 36, should give us every possible subgroup. Check! Also, need to draw the subgroup diagram.

28. This problem, like #23 requires finding all the subgroups of \mathbb{Z}_{20} . We did this in class on Monday!

29. We find that the only subgroups of \mathbb{Z}_{17} are $\{0\}$ and \mathbb{Z}_{17} (again, Theorem 6.14). The reason: since 17 is a prime number, the only positive divisors of 17 are 1 and 17.

34. Hint: Examples of such groups abound, so your solution may be different. However, $\langle \mathbb{Q}, + \rangle$ will work. The hard part is showing that \mathbb{Q} is not cyclic. By Theorem 6.10, we need to show that \mathbb{Q} is not isomorphic to \mathbb{Z} . This is most easily done by supposing that we have an isomorphism $\phi : \mathbb{Z} \rightarrow \mathbb{Q}$. Since such an isomorphism must be onto, there will be an integer $a \in \mathbb{Z}$ so that $\phi(a) = \frac{1}{2}$. Now derive a contradiction....

36. No such group exists. Here again, we use Theorem 6.10. If G is an infinite cyclic group having 4 generators. Since G is infinite cyclic, Theorem 6.10 says that $G \simeq \mathbb{Z}$. Suppose $\phi : G \rightarrow \mathbb{Z}$ is an isomorphism. If $a, b, c,$ and d are the four generators of G , show that each of their images $\phi(a), \phi(b), \phi(c), \phi(d)$ must be a generator of \mathbb{Z} . Why is this a contradiction?

41. Hint: The 12-th roots of unity are all of the form $e^{2\pi i/12}$. Use Theorem 6.14.

46. Hint: Write down the definitions of “ ab has finite order n ” and “ ba has finite order n .” We know that the first statement is true; use it to prove the second. Recall that every element of a finite group has finite order, as we proved in class.

49. Hint: We have seen examples of this in class.

50. Hint: The hint that book gives is a little cryptic but useful. Follow their advice by expanding the expression they give. You will make special use of the fact that a is the only element in G of order 2.

52. We know from Theorem 6.14 that the set of generators S of \mathbb{Z}_{p^r} will be

$$S = \{a \in \mathbb{Z} \mid 1 \leq a \leq p^r - 1, \gcd(a, p^r) = 1\}.$$

However, having $\gcd(a, p^r) = 1$ is the same thing as having the prime number p not dividing

a (since p itself has no positive divisors other than 1 and p). Thus

$$S = \{a \in \mathbb{Z}_{p^r} \mid p \nmid a\}.$$

Notice that $\mathbb{Z}_{p^r} - S$ (the elements of \mathbb{Z}_{p^r} that are not in S) are precisely the multiples of p in \mathbb{Z}_{p^r} . These multiples of p are $0, p, 2p, \dots, p^{r-1} \cdot p$, and there are p^{r-1} of them. Thus

$$|S| = |\mathbb{Z}_{p^r}| - |\mathbb{Z}_{p^r} - S| = p^r - p^{r-1} = p^{r-1}(p - 1).$$

Section 7, #4. Hint: Every subgroup of a cyclic group is cyclic by Theorem 6.6. So what is a cyclic generator for the subgroup of \mathbb{Z}_{36} generated by 12 and 30?

7. (a) a^3b ; (b) a^2 ; (c) a^2 .

10. Hint: Its multiplication table looks like one we've seen many times before.

11. There is an answer in the back of the book that is reasonable here. Though in practice it may not be so easy to check....