Math 627
Homework #3

September 20, 2006

Due Wednesday, September 27. All problems are to be turned in.

We recall some notation: $\mathcal{A} := \{f : \mathbb{Z}^+ \to \mathbb{C}\}$ is the ring of arithmetic functions, with binary operations addition $+$ and Dirichlet multiplication $\circ$. The zero element of $\mathcal{A}$ is the zero function 0 (the function that is identically zero). The multiplicative identity element is the function $\mathbb{I}$, defined by $\mathbb{I}(1) := 1$ and $\mathbb{I}(n) := 0$ if $n > 1$.

1. Show that $\mathcal{A}$ is an integral domain.

2. Define a function $D : \mathcal{A} \to \mathcal{A}$ in the following way:

   $$[D(f)](n) := f(n) \log n, \qquad n \geq 1.$$

   Here, and from now on, "log" will denote the natural logarithm. Show that D satisfies the Leibniz rule with respect to multiplication: for all $f, g \in \mathcal{A}$ we have

   $$D(f \circ g) = D(f) \circ g + f \circ D(g).$$

   Also, characterize all functions for which $D(f) = 0$.

3. A function $f \in \mathcal{A}$ is *multiplicative* if $f(1) = 1$ and if whenever $\gcd(m, n) = 1$ we have $f(mn) = f(m)f(n)$. (Note that this is a little more restrictive than the definition in Ireland and Rosen.) Let $\mathcal{M}$ denote the subset of multiplicative functions of $\mathcal{A}$. Show that $\mathcal{M}$ is a group under Dirichlet multiplication.

4. We say that a power series $\sum a_n x^n$ is a *rational function* if it is the Maclaurin series of a rational function. E.g., $\sum_{n=0}^{\infty} x^n = 1/(1-x)$.

   Given a positive prime number p and a function $f \in \mathcal{A}$, we define the formal power series

   $$B_p(f, x) := \sum_{j=0}^{\infty} f(p^j)x^j = f(1) + f(p)x + f(p^2)x^2 + \cdots .$$

   Let e, ν, σ, μ, $\phi \in \mathcal{A}$ denote the familiar arithmetic functions we have discussed in class ($e : \mathbb{Z}^+ \to \mathbb{C}$ is the function that is indentically 1: $e(n) = 1$ for all $n \in \mathbb{Z}^+$). For each of $f \in \{e, \nu, \sigma, \mu, \phi\}$ show that $B_p(f, x)$ is a rational function.

5. For all positive primes p and all $f, g \in \mathcal{A}$, show that $B_p(f \circ g, x) = B_p(f, x)B_p(g, x)$. (The product on the right denotes the usual multiplication of power series.)

6. Show that if $f, g \in \mathcal{M}$ then $B_p(f, x) = B_p(g, x)$ for all positive primes p if and only if $f = g$.

7. Let $f \in \mathcal{M}$ and $g \in \mathcal{A}$, and let p be a positive prime number. Suppose that (A) for all $e \geq 1$ we have

$$f(p^{e+1}) = f(p)f(p^e) - g(p)f(p^{e-1}).$$

Show that (B) we have

$$B_p(f,x) = \frac{1}{1 - f(p)x + g(p)x^2}.$$

Conversely show that (B) implies (A).