We recall some notation: \( A := \{ f : \mathbb{Z}^+ \to \mathbb{C} \} \) is the ring of arithmetic functions, with binary operations addition + and Dirichlet multiplication \( \circ \). The zero element of \( A \) is the zero function 0 (the function that is identically zero). The multiplicative identity element is the function \( I \), defined by \( I(1) := 1 \) and \( I(n) := 0 \) if \( n > 1 \).

1. Show that \( A \) is an integral domain.

2. Define a function \( D : A \to A \) in the following way:
   \[
   [D(f)](n) := f(n) \log n, \quad n \geq 1.
   \]
   Here, and from now on, “log” will denote the natural logarithm. Show that \( D \) satisfies the Leibniz rule with respect to multiplication: for all \( f, g \in A \) we have
   \[
   D(f \circ g) = D(f) \circ g + f \circ D(g).
   \]
   Also, characterize all functions for which \( D(f) = 0 \).

3. A function \( f \in A \) is multiplicative if \( f(1) = 1 \) and if whenever \( \gcd(m, n) = 1 \) we have \( f(mn) = f(m)f(n) \). (Note that this is a little more restrictive than the definition in Ireland and Rosen.) Let \( M \) denote the subset of multiplicative functions of \( A \). Show that \( M \) is a group under Dirichlet multiplication.

4. We say that a power series \( \sum a_n x^n \) is a rational function if it is the Maclaurin series of a rational function. E.g., \( \sum_{n=0}^{\infty} x^n = 1/(1 - x) \).
   Given a positive prime number \( p \) and a function \( f \in A \), we define the formal power series
   \[
   B_p(f, x) := \sum_{j=0}^{\infty} f(p^j) x^j = f(1) + f(p)x + f(p^2)x^2 + \cdots.
   \]
   Let \( e, \nu, \sigma, \mu, \phi \in A \) denote the familiar arithmetic functions we have discussed in class (\( e : \mathbb{Z}^+ \to \mathbb{C} \) is the function that is identically 1: \( e(n) = 1 \) for all \( n \in \mathbb{Z}^+ \)). For each of \( f \in \{ e, \nu, \sigma, \mu, \phi \} \) show that \( B_p(f, x) \) is a rational function.

5. For all positive primes \( p \) and all \( f, g \in A \), show that \( B_p(f \circ g, x) = B_p(f, x)B_p(g, x) \). (The product on the right denotes the usual multiplication of power series.)

6. Show that if \( f, g \in M \) then \( B_p(f, x) = B_p(g, x) \) for all positive primes \( p \) if and only if \( f = g \).
7. Let \( f \in M \) and \( g \in A \), and let \( p \) be a positive prime number. Suppose that (A) for all \( e \geq 1 \) we have
\[
f(p^{e+1}) = f(p)f(p^e) - g(p)f(p^e - 1).
\]
Show that (B) we have
\[
B_p(f, x) = \frac{1}{1 - f(p)x + g(p)x^2}.
\]
Conversely show that (B) implies (A).