Due Friday, October 6. All problems are to be turned in.

Problems from Ireland & Rosen: Ch. 2: 6, 7, 8.

We recall some notation: $A := \{ f : \mathbb{Z}^+ \to \mathbb{C} \}$ is the ring of arithmetic functions, with binary operations addition $+$ and Dirichlet multiplication $\circ$. The zero element of $A$ is the zero function $0$ (the function that is identically zero). The multiplicative identity element is the function $I$, defined by $I(1) := 1$ and $I(n) := 0$ if $n > 1$. The subset of $A$ of multiplicative functions is denoted $M$.

1. Let $P = \{ f \in A \mid \exists t \in \mathbb{R}, \exists c \in \mathbb{R}^+ \text{ so that } \forall n \in \mathbb{Z}^+, |f(n)| \leq cn^t \}$. Such functions have either polynomial growth or decay depending on whether $t$ is positive or negative.

For $f \in P$, we define the Dirichlet series of $f$ to be

$$L(f, s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$ 

(a) Show that $e, \mu, \phi, \sigma \in P$.

(b) Let $f \in P$. Let $t_f := \inf\{ t \in \mathbb{R} \mid \exists c \in \mathbb{R}^+ \text{ so that } \forall n \in \mathbb{Z}^+, |f(n)| \leq cn^t \}$. (As usual, “inf” means “greatest lower bound.” If no greatest lower bound exists, we set $t_f := -\infty$.) Show, for all $s \in \mathbb{R}$ with $s > t_f + 1$, that $L(f, s)$ is absolutely convergent.

(c) If $L(f, s)$ and $L(g, s)$ are absolutely convergent at $s \in \mathbb{R}^+$, show that $L(f \circ g, s)$ is also absolutely convergent.

(d) If $L(f, s)$ and $L(g, s)$ are absolutely convergent at $s \in \mathbb{R}^+$, show that $L(f, s)L(g, s) = L(f \circ g, s)$.

The upshot is that the product of two Dirichlet series is also a Dirichlet series and that the coefficients of the product are obtained from the Dirichlet product of the coefficients of the two factors.

2. Let $f \in M \cap P$. Suppose that $s > t_f + 1$.

(a) For a positive prime $p$, show that the series $B_p(f, p^{-s})$ is absolutely convergent, where $B_p(f, x)$ is the series defined in HW#3.

(b) Show that

$$L(f, s) = \prod_p B_p(f, p^{-s}),$$

where the product runs over all positive prime numbers $p$. Such a product is called an Euler product. (FYI: It can be shown fairly easily that the converse is also true—that if the Dirichlet series of a function $f \in P$ can be expressed as an Euler product then it must be multiplicative.)
3. In this problem we refer to the results of problem 4 in HW#3.

(a) Express \( L(e, s), L(\nu, s), L(\sigma, s), L(\mu, s), L(\phi, s) \) as Euler products whose factors are rational functions of \( p^{-s} \). Usually one writes \( \zeta(s) = L(e, s) = \sum \frac{1}{n^s} \), since this function is also called the Riemann zeta function.

(b) Verify that \( \frac{1}{\zeta(s)} = L(\mu, s) \).

(c) Verify that \( \zeta(s)^2 = L(\nu, s) \).

(d) Verify that \( \zeta(s) \zeta(s - 1) = L(\sigma, s) \).

(e) Verify that \( \frac{\zeta(s - 1)}{\zeta(s)} = L(\phi, s) \).

4. Consider the function \( D : \mathcal{A} \to \mathcal{A} \) defined in problem 2 of HW#3. Show that \( D(\mathcal{P}) \subseteq \mathcal{P} \). For \( f \in \mathcal{P} \), what function is \( L(D(f), s) \)?

5. The function prime counting function \( \pi : \mathbb{R}^+ \to \mathbb{Z} \) is a step function and hence is piece-wise continuous and integrable. Show that for \( x \geq 2 \),
\[
\int_2^x \frac{\pi(t)}{t} \, dt = \pi(x) \log x - \theta(x).
\]

One way to do this is to list all of the primes up to \( x \) in order as \( p_1 = 2, p_2 = 3, \ldots, p_n \leq x \). For each \( i \), on the interval \( p_i \leq t < p_{i+1} \), what is the value of \( \pi(t) \)? Use the answer to this question when you break up the integral as
\[
\int_2^x \frac{\pi(t)}{t} \, dt = \sum_{i=1}^{n-1} \int_{p_i}^{p_{i+1}} \frac{\pi(t)}{t} \, dt + \int_{p_n}^x \frac{\pi(t)}{t} \, dt.
\]

Now compute!