

Math 627
Homework #4

September 27, 2006

Due Friday, October 6. All problems are to be turned in.

Problems from Ireland & Rosen: Ch. 2: 6, 7, 8.

We recall some notation: $\mathcal{A} := \{f : \mathbb{Z}^+ \rightarrow \mathbb{C}\}$ is the ring of arithmetic functions, with binary operations addition $+$ and Dirichlet multiplication \circ . The zero element of \mathcal{A} is the zero function 0 (the function that is identically zero). The multiplicative identity element is the function \mathbb{I} , defined by $\mathbb{I}(1) := 1$ and $\mathbb{I}(n) := 0$ if $n > 1$. The subset of \mathcal{A} of multiplicative functions is denoted \mathcal{M} .

1. Let $\mathcal{P} = \{f \in \mathcal{A} \mid \exists t \in \mathbb{R}, \exists c \in \mathbb{R}^+ \text{ so that } \forall n \in \mathbb{Z}^+, |f(n)| \leq cn^t\}$. Such functions have either polynomial growth or decay depending on whether t is positive or negative.

For $f \in \mathcal{P}$, we define the *Dirichlet series of f* to be

$$L(f, s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

- (a) Show that $e, \mu, \phi, \sigma \in \mathcal{P}$.
- (b) Let $f \in \mathcal{P}$. Let $t_f := \inf\{t \in \mathbb{R} \mid \exists c \in \mathbb{R}^+ \text{ so that } \forall n \in \mathbb{Z}^+, |f(n)| \leq cn^t\}$. (As usual, “inf” means “greatest lower bound.” If no greatest lower bound exists, we set $t_f := -\infty$.) Show, for all $s \in \mathbb{R}$ with $s > t_f + 1$, that $L(f, s)$ is absolutely convergent.
- (c) If $L(f, s)$ and $L(g, s)$ are absolutely convergent at $s \in \mathbb{R}^+$, show that $L(f \circ g, s)$ is also absolutely convergent.
- (d) If $L(f, s)$ and $L(g, s)$ are absolutely convergent at $s \in \mathbb{R}^+$, show that $L(f, s)L(g, s) = L(f \circ g, s)$.

The upshot is that the product of two Dirichlet series is also a Dirichlet series and that the coefficients of the product are obtained from the Dirichlet product of the coefficients of the two factors.

2. Let $f \in \mathcal{M} \cap \mathcal{P}$. Suppose that $s > t_f + 1$.

- (a) For a positive prime p , show that the series $B_p(f, p^{-s})$ is absolutely convergent, where $B_p(f, x)$ is the series defined in HW#3.
- (b) Show that

$$L(f, s) = \prod_p B_p(f, p^{-s}),$$

where the product runs over all positive prime numbers p . Such a product is called an *Euler product*. (FYI: It can be shown fairly easily that the converse is also true—that if the Dirichlet series of a function $f \in \mathcal{P}$ can be expressed as an Euler product then it must be multiplicative.)

3. In this problem we refer to the results of problem 4 in HW#3.
- Express $L(e, s)$, $L(\nu, s)$, $L(\sigma, s)$, $L(\mu, s)$, $L(\phi, s)$ as Euler products whose factors are rational functions of p^{-s} . Usually one writes $\zeta(s) = L(e, s) = \sum \frac{1}{n^s}$, since this function is also called the *Riemann zeta function*.
 - Verify that $1/\zeta(s) = L(\mu, s)$.
 - Verify that $\zeta(s)^2 = L(\nu, s)$.
 - Verify that $\zeta(s)\zeta(s-1) = L(\sigma, s)$.
 - Verify that $\zeta(s-1)/\zeta(s) = L(\phi, s)$.
4. Consider the function $D : \mathcal{A} \rightarrow \mathcal{A}$ defined in problem 2 of HW#3. Show that $D(\mathcal{P}) \subseteq \mathcal{P}$. For $f \in \mathcal{P}$, what function is $L(D(f), s)$?
5. The function prime counting function $\pi : \mathbb{R}^+ \rightarrow \mathbb{Z}$ is a step function and hence is piece-wise continuous and integrable. Show that for $x \geq 2$,

$$\int_2^x \frac{\pi(t)}{t} dt = \pi(x) \log x - \theta(x).$$

One way to do this is to list all of the primes up to x in order as $p_1 = 2, p_2 = 3, \dots, p_n \leq x$. For each i , on the interval $p_i \leq t < p_{i+1}$, what is the value of $\pi(t)$? Use the answer to this question when you break up the integral as

$$\int_2^x \frac{\pi(t)}{t} dt = \sum_{i=1}^{n-1} \int_{p_i}^{p_{i+1}} \frac{\pi(t)}{t} dt + \int_{p_n}^x \frac{\pi(t)}{t} dt.$$

Now compute!