Due Friday, February 9.

1. Let $R$ be a commutative ring and $R[x]$ be the polynomial ring in $t$ over $R$. For $f(x) = a_nx^n + \cdots + a_1x + a_0$, define the derivative of $f$ in the usual way:

$$f'(x) := na_nx^{n-1} + \cdots + a_1 \in R[x].$$

(a) Verify that the following identities hold for all $f, g \in R[x]$, and $c \in R$:

$$(f + g)' = f' + g',$$

$$(fg)' = fg' + f'g,$$

$$(cf)' = c(f'),$$

$$(f^m)' = mf^{m-1}f'.$$

(b) Let $f \in R[x]$ and $a \in R$. Prove that if $(x-a)^2 \mid f(x)$ in $R[x]$, then $(x-a) \mid f'(x)$.

(c) Let $f \in R[x]$ and $a \in R$. Prove that if $(x-a) \mid f(x)$ and $(x-a) \mid f'(x)$ in $R[x]$, then $(x-a)^2 \mid f(x)$.

(d) Let $F$ be a field and let $f \in F[x]$, deg $f \geq 1$. Prove that there exists an irreducible polynomial $p \in F[x]$, deg $p \geq 1$, such that $p^2 \mid f$ in $F[x]$ if and only if the greatest common divisor of $f$ and $f'$ in $F[x]$ has positive degree.

2. Consider the following polynomials:

$$f_1 = x^2 + 3x + 1, \quad f_2 = x^3 - 3x + 1, \quad f_3 = x^5 - 2, \quad f_4 = x^4 + 4.$$

(a) Which of $f_1, f_2, f_3, f_4$ is irreducible in $\mathbb{Q}[x]$? If one is not, provide a complete factorization.

(b) For $i = 1, 2, 3, 4$, let $E_i$ be the field obtained by adjoining all of the roots of $f_i$ in $\mathbb{C}$ to $\mathbb{Q}$. Show that $[E_1 : \mathbb{Q}] = 2$ and $[E_2 : \mathbb{Q}] = 3$. What are $[E_3 : \mathbb{Q}]$ and $[E_4 : \mathbb{Q}]$? One thing that we will prove on Monday is that if $F \subseteq E \subseteq K$ are fields, then $[K : F] = [K : E][E : F]$. Feel free to use this in the meantime. (Apology/warning: showing that $[E_2 : \mathbb{Q}] = 3$ is a little hard.)