

Math 654
Homework #1

February 2, 2007

Due Friday, February 9.

1. Let R be a commutative ring and $R[x]$ be the polynomial ring in x over R . For $f(x) = a_n x^n + \cdots + a_1 x + a_0$, define the *derivative* of f in the usual way:

$$f'(x) := na_n x^{n-1} + \cdots + a_1 \in R[x].$$

- (a) Verify that the following identities hold for all $f, g \in R[x]$, and $c \in R$:

$$\begin{aligned} (f+g)' &= f' + g', & (cf)' &= c(f'), \\ (fg)' &= fg' + f'g, & (f^m)' &= mf^{m-1}f'. \end{aligned}$$

- (b) Let $f \in R[x]$ and $a \in R$. Prove that if $(x-a)^2 \mid f(x)$ in $R[x]$, then $(x-a) \mid f'(x)$.
- (c) Let $f \in R[x]$ and $a \in R$. Prove that if $(x-a) \mid f(x)$ and $(x-a) \mid f'(x)$ in $R[x]$, then $(x-a)^2 \mid f(x)$.
- (d) Let F be a field and let $f \in F[x]$, $\deg f \geq 1$. Prove that there exists an irreducible polynomial $p \in F[x]$, $\deg p \geq 1$, such that $p^2 \mid f$ in $F[x]$ if and only if the greatest common divisor of f and f' in $F[x]$ has positive degree.

2. Consider the following polynomials:

$$f_1 = x^2 + 3x + 1, \quad f_2 = x^3 - 3x + 1, \quad f_3 = x^5 - 2, \quad f_4 = x^4 + 4.$$

- (a) Which of f_1, f_2, f_3, f_4 is irreducible in $\mathbb{Q}[x]$? If one is not, provide a complete factorization.
- (b) For $i = 1, 2, 3, 4$, let E_i be the field obtained by adjoining all of the roots of f_i in \mathbb{C} to \mathbb{Q} . Show that $[E_1 : \mathbb{Q}] = 2$ and $[E_2 : \mathbb{Q}] = 3$. What are $[E_3 : \mathbb{Q}]$ and $[E_4 : \mathbb{Q}]$? One thing that we will prove on Monday is that if $F \subseteq E \subseteq K$ are fields, then $[K : F] = [K : E][E : F]$. Feel free to use this in the meantime. (Apology/warning: showing that $[E_2 : \mathbb{Q}] = 3$ is a little hard.)