Math 654
Homework #2

February 23, 2007

Due Friday, March 2.

Recall: A finite extension E/k is *Galois* if it is separable and if E is the splitting field of some polynomial $f \in k[x]$.

1. Let k be a field, and let E/k be a finite extension. Let N be a normal closure of E/k. Let

   $$\mathrm{Emb}(E/k) := \{\sigma : E \to N \mid \sigma \text{ is a field homomorphism and } \sigma|_k = \mathrm{id}_k\}.$$

   Finally suppose E/k is separable.

   (a) Show that $\#\,\mathrm{Emb}(E/k) = [E : k]$.

   (b) Prove that E/k is Galois if and only if $\mathrm{Emb}(E/k) = \mathrm{Gal}(E/k)$. (Usually one says that "E/k is Galois if and only if every embedding of E/k induces and automorphism of E/k.")

2. Let $m \geq 1$ be an integer, let $\zeta = e^{2\pi i/m} \in \mathbb{C}$.

   (a) Show that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois.

   (b) Let $G := \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. Define a function $\Psi : G \to \mathbb{Z}/m\mathbb{Z}$ in the following way. For $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ define $\Psi(\sigma)$ so that

   $$\sigma(\zeta) = \zeta^{\Psi(\sigma)}.$$

   Show that $\Psi(\sigma)$ only depends on its class modulo m, and so is well-defined.

   (c) Show that the image of Ψ is contained in $(\mathbb{Z}/m\mathbb{Z})^\times$ and that the induced function $\Psi : G \to (\mathbb{Z}/m\mathbb{Z})^\times$ is an injective group homomorphism.

   (d) Suppose that m is prime. Prove that Ψ is an isomorphism. (In fact, Ψ is an isomorphism for all integers $m \geq 1$, but this takes some doing.)

3. Let $E = \mathbb{Q}(\sqrt[5]{2}, \zeta)$, where $\sqrt[5]{2}$ is the real 5-th root of 2 and $\zeta = e^{2\pi i/5}$.

   (a) Show that $E/\mathbb{Q}$ is Galois. What is $[E : \mathbb{Q}]$?

   (b) Let $G = \mathrm{Gal}(E/\mathbb{Q})$. We first define two functions $\alpha : G \to \mathbb{F}_5^\times$ and $\beta : G \to \mathbb{F}_5$. Since $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois by the previous problem, we have a natural surjective homomorphism

   $$R : G \to \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}),$$

   where $R(\sigma) = \sigma|_{\mathbb{Q}(\zeta)}$. Let $\alpha := \Psi \circ R : G \to \mathbb{F}_5^\times$, where Ψ is defined as in problem 2. Now for $\sigma \in G$, we define $\beta(\sigma)$ by requiring that

   $$\sigma(\sqrt[5]{2}) = \zeta^{\beta(\sigma)}\sqrt[5]{2}.$$

   Here is the problem: Let $\mathrm{GL}_2(\mathbb{F}_5)$ be the group of 2×2 matrices with entries in $\mathbb{F}_5$ that are invertible. Show that the assignment

   $$\sigma \mapsto \begin{bmatrix} \alpha(\sigma) & \beta(\sigma) \\ 0 & 1 \end{bmatrix} : G \to \mathrm{GL}_2(\mathbb{F}_5)$$

   is an injective group homomorphism. What is its image?