Math 662
Homework #1

February 24, 2005

Due Friday, March 4.
Do 2 of sets A–D below. Problems marked with a '*' are for extra credit.

A. Koblitz, §I.6: 3, 4, 5, 6

B. Koblitz, §I.7: 3, 5; §I.8: 9 (a)–(e), (f)*, (g)*

C. Koblitz, §I.6: 1, 2; Silverman, p. 167: 6.6.

To do part (b) of problem 6.6, you may want to prove the following: Suppose Λ, Θ are lattices in $\mathbb{C}$, and suppose there is a $c \in \mathbb{C}^\times$ so that $g_2(\Lambda) = c^{-4} g_2(\Theta)$ and $g_3(\Lambda) = c^{-6} g_3(\Theta)$. Then there are isomorphisms

$$\mathbb{C}/\Lambda \xrightarrow{\Phi_\Lambda} E_\Lambda \xrightarrow{f} E_\Theta \xrightarrow{\Phi_\Theta^{-1}} \mathbb{C}/\Theta,$$

where for a point $P = (x, y) \in E_\Lambda$, we set $f(P) := (c^{-2}x, c^{-3}y)$. Why f is a group isomorphism? Then use the following fact: it turns out that because $\mathbb{C}$ is a universal covering space for $\mathbb{C}/\Lambda$ that there is an entire function $F : \mathbb{C} \to \mathbb{C}$ so that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Lambda & \xrightarrow{f} & \mathbb{C}/\Theta \end{array}$$

commutes. (The vertical maps are the canonical projection maps.) Prove that $F(z) = cz$ for all $z \in \mathbb{C}$. This will help greatly in solving problem 6.6.

D. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$, and let $\wp(z)$ be its associated Weierstrass $\wp$-function. In class we showed that the function

$$g(z) := \wp(2z) + 2\wp(z) - \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$$

has at worst simple poles at each point in $\frac{1}{2}\Lambda$ and no other poles.

(1) Show that in fact g is identically 0.

(2) Use this formula to prove that $\Phi_\Lambda(2z) = [2]_{E_\Lambda}(\Phi_\Lambda(z))$, where $\Phi_\Lambda : \mathbb{C} \to E_\Lambda$ is the function defined in class sending $z \notin \Lambda$ to $(\wp(z), \wp'(z)) \in E_\Lambda$.

(3) Use the identity

$$\wp(z + w) = -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2$$

to show the following: Suppose that z, $w \in \mathbb{C}$ and z, w, $z \pm w$ are not in Λ. Then the x-coordinate of $\Phi(z + w)$ is the x-coordinate of $\Phi(z) +_{E_\Lambda} \Phi(w)$.