Math 662
Homework #2

March 24, 2005

Due Friday, April 8.
Do 2 of sets A–D below.

A. Koblitz, §III.2: 1, 3, 4

B. Problems on Bernoulli numbers:

(i) Show that the Bernoulli numbers B_k satisfy the recursion:

$$(m+1)B_m = -\sum_{k=0}^{m-1} \binom{m+1}{k} B_k.$$

(ii) For a positive integer m, let $S_m(n) = 1^m + 2^m + 3^m + \cdots + (n-1)^m$. Show that

$$(m+1)S_m(n) = \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

(Hint: Consider the power series expansion of $1 + e^x + e^{2x} + \cdots + e^{(n-1)x}$.) Use this to find closed formulas for $S_m(n)$ for $1 \leq m \leq 6$.

(iii) Let p be an odd prime. Show that for all $m \geq 1$ that pB_m has no factors of p in its reduced denominator. Show that $S_{p-1}(p) \equiv pB_{p-1} \pmod{p}$. Show that if $(p-1) \mid m$ then $S_m(p) \equiv -1 \pmod{p}$.

(iv) Let p be an odd prime. Show that, as a power series in q, $E_{p-1} \equiv 1 \pmod{p}$.

C. Koblitz, §III.2: 7, 8; also the following

(v) Let f be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Show that

$$C(f) := (k+1)\left(\frac{df}{dz}\right)^2 - kf\frac{d^2 f}{dz^2}$$

is a cusp form of weight $2k+4$. Show that

$$C(G_4) = \frac{1}{2^4 3^3 5^2 \pi^2}\Delta.$$

Show that $C(G_6) = cG_4\Delta$ for some $c \in \mathbb{C}$. What is c?

D. Koblitz, §III.2: 9; also the following

(vi) Look up the statement of Euler's pentagonal number theorem and explain the comment at the end of problem 9.