

Math 662
Homework #3

April 23, 2005

Due Tuesday, May 2.

Notation: Let k be an even integer greater than 4 throughout.

1. Let $k \geq 4$ be even, and let E_k be the usual normalized Eisenstein series of weight k for $\mathrm{SL}_2(\mathbb{Z})$.
 - (a) Prove that E_k is an eigenform for $T_k(n)$ for each $n \geq 1$.
 - (b) Let $f \in M_k$. Suppose that f is not a cusp form but that f is an eigenfunction for $T_k(n)$ for each $n \geq 1$. Show that f is a constant multiple of E_k .
2. For any function $f : \mathbb{H} \rightarrow \mathbb{C}$ and matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ with $\det \alpha > 0$, define a new function $f|[\alpha]_k : \mathbb{H} \rightarrow \mathbb{C}$ by

$$(f|[\alpha]_k)(z) = (\det \alpha)^{k/2} (cz + d)^{-k} f(\alpha z).$$

- (a) Show that if $f \in M_k$ then $f|[\gamma]_k = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
- (b) Prove that the operation of $T_k(n)$ on a modular function f of weight k is given by the formula

$$T_k(n)f = n^{k/2-1} \sum_{\alpha \in S_n} f|[\alpha]_k,$$

$$\text{where } S_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad = n, a, d > 0, 0 \leq b < d \right\}.$$

3. Let $k \geq 4$ be even. For $f, g \in S_k$, we define the Petersson inner product,

$$\langle f, g \rangle := \iint_{\mathcal{F}} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2},$$

where $\mathcal{F} = \{z \in \mathbb{H} : |\mathrm{Re} z| \leq \frac{1}{2}, |z| \geq 1\}$ is the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ and $z = x + iy$.

- (a) Prove that the integral converges.
- (b) Prove that $\langle \cdot, \cdot \rangle$ is a Hermitian inner product on S_k .
- (c) Let $\omega(f, g)$ be the integrand in the definition of $\langle \cdot, \cdot \rangle$. For any matrix $\alpha \in \mathrm{GL}_2(\mathbb{R})$ with $\det \alpha > 0$, show that

$$\omega(f, g) \circ \alpha = \omega(f|[\alpha]_k, g|[\alpha]_k).$$

4. Koblitz: §III.5: 6

5. Koblitz: §III.3: Do 8 on your own (not to turn in). Then turn in 10, 11* (bonus).