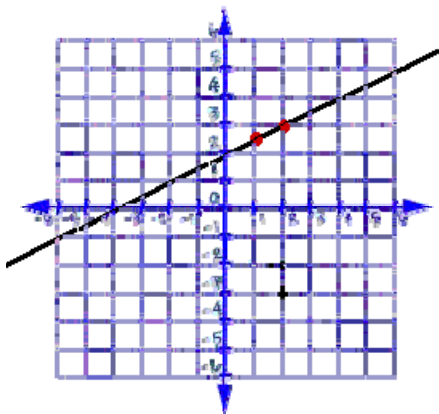


Math 142 Lecture Notes Section 3.1 – Introduction to Limits

★ Introduction to Calculus:

Four major problems of 17th century mathematicians were:

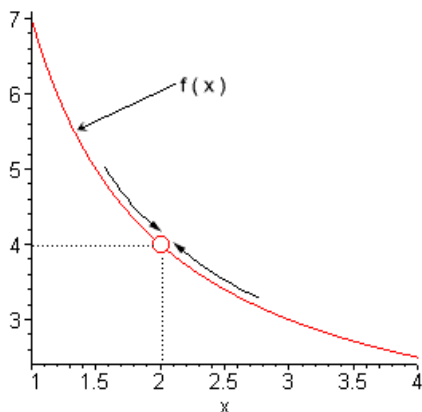
- 1) calculating instantaneous velocity of a moving body
- 2) find the slope of the line tangent to a curve at a given point
- 3) find the maximum or minimum value of a function over some given interval
- 4) finding the area under a curve.



★ Finding Values of a Function from Its Graph:

Without being given the function, we can determine the points on the graph of $f(x)$ shown above.

- 1) $f(2)=$
- 2) $f(-2)=$
- 3) $f(-4)=$
- 4) $f(6)=$



Look at the graph of $f(x)$ above. The function is not defined at 2. We can tell this by the hole in the graph. Again we can determine values for the function at other points.

$$\begin{aligned} f(3) &= 3 \\ f(1.5) &= 5 \\ f(2) &= \text{is not defined} \end{aligned}$$

We can look at the function as the x values from the right get closer, and closer to 2, and see what is happening to the y values.

$$\text{Mathematically this is written } \lim_{x \rightarrow 2^+} f(x) = 4$$

We can look at the function as the x values from the left get closer, and closer to 2, and see what is happening to the y values.

$$\text{Mathematically this is written } \lim_{x \rightarrow 2^-} f(x) = 4$$

As we look at functions, sometimes the limit from the left differs from the limit from the right, and in those cases, the limit at the point does not exist. If the two values are the same, then the limit at that point is the same as the limit from the left or the limit from the right.

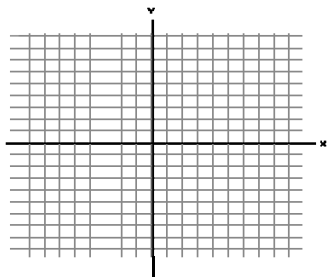
$$\text{Mathematically we would write that as } \lim_{x \rightarrow 2} f(x) = 4$$

Limit: $\lim_{x \rightarrow c} f(x) = L$ if the value of $f(x)$ is close to L , whenever x is close to c .
(meaning from either side of c , the y value is getting close to L)
The limit at c has nothing to do with the value of the function at c . It may not be defined at that point.

Example:

Graph $f(x) = \frac{|x-4|}{x-4}$

From the graph determine the limit of $f(x)$ as $x \rightarrow 4$



Theorem 1: The Existence of a Limit

For a limit to exist, the limit from the left and the limit from the right must exist and be equal. Mathematically, we write,

$$\lim_{x \rightarrow c} f(x) = L \text{ if and only if } \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

★ **Analyzing Limits Graphically.**

Graph: $f(x) = \begin{cases} -|x-3|, & x < 5 \\ -0.5x + 5, & x \geq 5 \end{cases}$

★ **Properties of Limits:**

Let f and g be two functions with $\lim_{x \rightarrow c} f(x) = L$, $\lim_{x \rightarrow c} g(x) = M$

1. $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$
2. $\lim_{x \rightarrow c} [f(x) - g(x)] = L - M$
3. $\lim_{x \rightarrow c} k \cdot f(x) = k \cdot \lim_{x \rightarrow c} f(x) = k \cdot L$ for any constant k
4. $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = [\lim_{x \rightarrow c} f(x)][\lim_{x \rightarrow c} g(x)] = LM$
5. $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{L}{M}$ if $M \neq 0$
6. $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)} = \sqrt[n]{L}$, $L > 0$ for n even

Find each of the following limits:

1. $\lim_{x \rightarrow 4} (x^2 - 2x)$
2. $\lim_{x \rightarrow -3} (4 - 2x)$
3. $\lim_{x \rightarrow a} (9 - x^2)$

★ **Limit of a polynomial or rational function at a point:**

To find the limit as x approaches c , plug in c .

If plugging in c produces a zero in the denominator, factor, reduce, and plug in c again.

1. $\lim_{x \rightarrow -1} (x^2 - 5x + 4)$
2. $\lim_{x \rightarrow 3} (9 - 5x)$
3. $\lim_{x \rightarrow 0} \sqrt{x^2 + 4}$
4. $\lim_{x \rightarrow -5} \frac{2x}{x + 5}$
5. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + x - 6}$
6. $\lim_{x \rightarrow -1} \sqrt{2x^2 + 1}$
7. $f(x) = \begin{cases} 4x - 6, & x < 3 \\ x^2 + 1, & x \geq 3 \end{cases}$ (a) $\lim_{x \rightarrow 3^+} f(x)$, (b) $\lim_{x \rightarrow 3^-} f(x)$
- (c) $\lim_{x \rightarrow 3} f(x)$, (d) $f(3)$

★ **Indeterminate Form:** If the limit of $f(x)$ as x approaches c is zero, and the limit of $g(x)$ as x approaches c is zero, then the limit as x approaches c of the quotient of $f(x)$ and $g(x)$ is indeterminate.

Example 1: $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

Example 2: $\lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$

Example 3: $\lim_{x \rightarrow 1} \frac{(x - 1)^2}{x^2 - 1}$

★ **Limit of a Difference Quotient:**

Find $\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$ for $f(x) = 8x - 12$

Find $\lim_{h \rightarrow 0} \frac{f(4 + h) - f(4)}{h}$ for $f(x) = \sqrt{x}$