

Math142 Lecture Notes

5.2 - Second Derivative and Graphs

First Derivative	Second Derivative	n^{th} derivative
$f'(x)$	$f''(x)$	$f^{(n)}(x)$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^{(n)}y}{dx^{(n)}}$

Example 1: Find the first three derivatives for $f(x) = 2x^3 - 6x^2 - 48x + 20$.

Example 2: Discuss the difference in the shapes of the graphs of $f(x) = x^2$ and $g(x) = \sqrt{x}$ on the interval $[0, \infty)$.

Clues to Graphing Functions

If $f(x) > 0$, it's graph lies above the x-axis.

If $f'(x) > 0$, the graph of $f(x)$ is increasing.

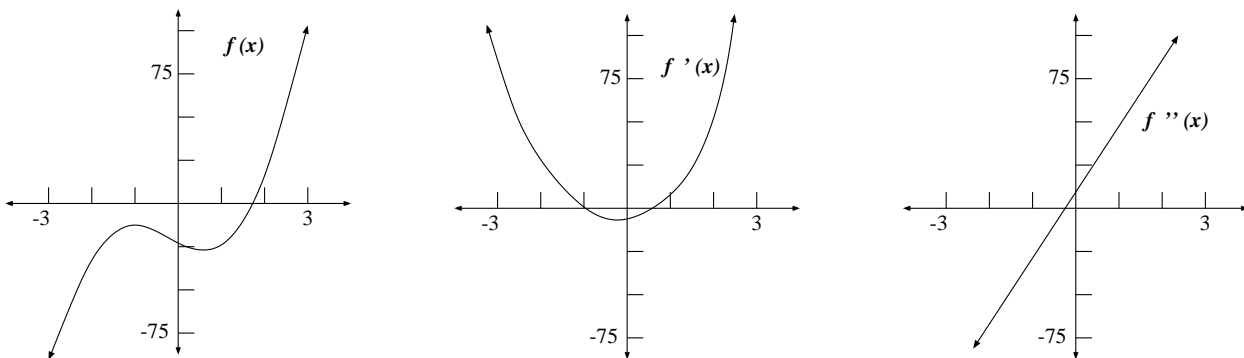
If $f''(x) > 0$, the graph of $f(x)$ is concave up.

Example 3: Given that $f(x) = x^3 - 6x^2 + 5$, determine the intervals where the graph:

a. is concave upward;

b. is concave downward.

Example 4: Given $f''(x) = 30x + 8$, find where $f''(x) = 0$ then find the intervals where $f'(x)$ is increasing and decreasing.



Concavity

On an open interval (a, b) where f is differentiable:

1. If $f'(x)$ is increasing ($f''(x) > 0$) then the graph of $f(x)$ is concave up.
2. If $f'(x)$ is decreasing ($f''(x) < 0$) then the graph of $f(x)$ is concave down.

Example 5: Determine the intervals where the graph of $f(x) = x^3 + 3x^2 - 4$ is concave up and concave down.

Second Derivative Test (determines inflection values and concavity)

1. Find all inflection values. These are the values $x = d$ where $f''(d) = 0$ or $f''(d)$ is undefined.
2. Place these values on a number line and use the second derivative to generate a sign chart.
3. The point $(d, f(d))$ is an inflection point if $f''(x)$ changes sign at $x = d$ and if $x = d$ is in the domain of $f(x)$.

Example 6: Find the inflection points of $f(x)$ if $f''(x) = (x + 2)(x - 1)x^2$.

Example 7: The Chug-a-Mug Company has determined that its cost, in hundreds of dollars, for producing x items of its best selling product is given by $C(x) = x^3 - 6x^2 + 15x$.

(a) Determine $C(5)$ and $C'(5)$ and interpret each.

(b) Determine the intervals where $C'(x)$ is increasing and where it is decreasing. Determine the relative minimum for the marginal cost function.

(c) Determine the inflection point for the graph of C .

Point of Diminishing Returns: The company expects increased dollars spent on advertising to increase sales. The value of x where the **rate of change of sales** changes from increasing to decreasing is called the point of **diminishing returns**.

Example 8: A T-shirt manufacturer is planning to expand its workforce. It estimates that the number of T-shirts produced by hiring x new workers is given by

$$T(x) = -0.25x^4 + 5x^3, 0 \leq x \leq 15$$

When is the rate of change of T-shirt production increasing and when is it decreasing? what is the point of diminishing returns and the maximum rate of change of T-shirt production?

Source: p310 Calculus for Business, Economics, Life Sciences, and Social Sciences by Barnett, Ziegler, Byleen, 11th edition

Curve Sketching

A Graphing Strategy

Analyze $f(x)$

- Find the domain of the function.
- Find the intercepts.
- Find the asymptotes.

Analyze $f'(x)$

- Find the zeros of $f'(x)$
- Construct a sign chart for $f'(x)$, and determine where the function is \nearrow and/or \searrow .
- Find the local max/min.

Analyze $f''(x)$.

- Find the zeros of $f''(x)$.
- Using a sign chart, determine where the graph of the function is \cup or \cap .
- Find any points of inflection.

Sketch the graph of the function.

Example 9: Analyze the graph of $f(x) = x^4 + 2x^3$