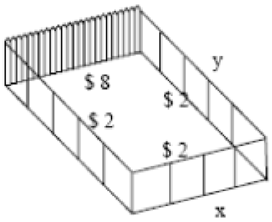
Math142 Lecture Notes
5.6 - Optimization

Today let's look at word problems which involve finding maximums or minimums. Note: the function to be maximized (or minimized) is not usually provided, so make your first goal to write the function for which you must find the max/min. For example: if the problem asks you to find the price which **maximizes profit**, write the function for profit, $P(x) = \ldots$. If the problem asks you to find the width of the garden which **minimizes cost**, write the function for cost, $C(x) = \ldots$.

Example 1: A homeowner has \$320 to spend on building a fence around a rectangular garden. Three sides of the fence will be constructed with wire fencing at a cost of \$2 per linear foot. The fourth side is to be constructed with wood fencing at a cost of \$8 per linear foot. Find the dimensions and the area of the largest garden that can be enclosed with \$320 worth of fencing. Source: Calculus for Business, Economics, Life Sciences, and Social Sciences, Barnett, p.344

Step 1: **Sketch a picture or construct a table to represent the problem and define the variables to represent the unknowns.**



x = width in feet

y = length in feet

$$320 = 2y + 2x + 2y + 8x$$

$$320 = 10x + 4y$$

$$320 - 10x = 4y$$

$$80 - \frac{10}{4}x = y$$

$$80 - 2.5x = y$$

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Step 2: Formulate an equation that will represent the quantity to be maximized or minimized and determine the interval (reasonable domain) over which the function is to be optimized.

$A = x \cdot y$

domain $0 < x < 32$

$A = x(80 - 2.5x) = 80x - 2.5x^2$

Step 3: Find the absolute extrema, checking the endpoints as well as the critical values.

$A' = 80 - 5x = 0$

$80 = 5x$

$16 = x$

Step 4: Answer the question posed in the problem.

$A'' = -5$ $\therefore$ $x = 16$ is the location of a maximum

"Find the dimensions"

$y = 80 - 2.5x$

$y = 80 - 2.5(16)$

$y = 40$

$\boxed{16' \times 40'}$

"Find the area"

$A = (16)(40)$

$A = \boxed{640 \text{ sq. ft.}}$

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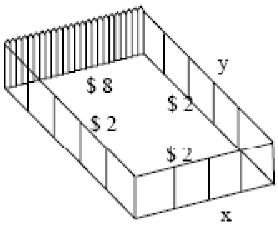
Example 2: Suppose the homeowner from example 1, decides he needs 1000 sq.ft. for his garden. What is the minimum cost of building a fence that will enclose a garden with area 1,000 square feet? What are the dimensions of this garden? Assume that the cost of fencing remains unchanged.

Step 1: Sketch a picture or construct a table to represent the problem and define the variables to represent the unknowns.



$x \cdot y = 1000$

$y = \frac{1000}{x}$

$C = 2y + 2x + 2y + 8x$

$C = 10x + 4y$

$C = 10x + 4\left(\frac{1000}{x}\right)$

$C = 10x + 4000x^{-1}$

Step 2: Formulate an equation that will represent the quantity to be maximized or minimized and determine the interval (reasonable domain) over which the function is to be optimized.

domain $x > 0$

$C = 4000x^{-1} + 10x$

$C' = -4000x^{-2} + 10 = 0$

$10 = \frac{4000}{x^2}$

$10x^2 = 4000$

$x^2 = 400$

$x = 20$

$\boxed{\text{Step 3:}}$ Find the absolute extrema, checking the endpoints as well as the critical values.

$$C'' = 8000x^{-3} = \frac{8000}{x^3}$$

$$C''(20) = + \quad \therefore \text{concave up}$$

$x = 20$ produces minimum cost

$\boxed{\text{Step 4:}}$ Answer the question posed in the problem.

"What are the dimensions"

$$y = \frac{1000}{x}$$

$$y = \frac{1000}{20} = 50$$

$$\boxed{\begin{array}{l} x = 20\text{ ft.} \\ y = 50\text{ ft.} \end{array}}$$

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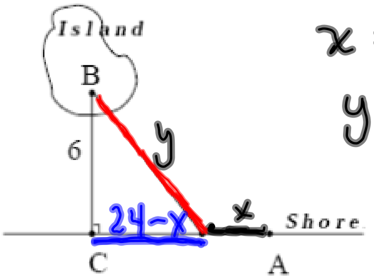
Example 3: A company wishes to run a utility cable from point A on the shore, to an installation at point B on the island. The distance from point A to point C is 24 miles. The island is 6 miles from the shore. It costs \$400 per mile to run the cable on land and \$500 per mile underwater. Assume that the cable starts at A and runs along the shoreline, then angles and runs underwater to the island. Find the point at which the line should begin to angle in order to yield the minimum total cost.

Step 1: Sketch a picture or construct a table to represent the problem and define the variables to represent the unknowns.



x = # miles parallel to shore

y = # miles underwater

Pythagorean Theorem

$$6^2 + (24-x)^2 = y^2$$

$$36 + 576 - 48x + x^2 = y^2$$

Step 2: Formulate an equation that will represent the quantity to be maximized or minimized and determine the interval (reasonable domain) over which the function is to be optimized.

domain

$0 \le x \le 24$

$$C = 400x + 500\sqrt{x^2 - 48x + 612}$$

$$C = 400x + 500(x^2 - 48x + 612)^{1/2}$$

$$C' = 400 + 250(x^2 - 48x + 612)^{-1/2}(2x - 48)$$

$$C' = 400 + \frac{250(2x - 48)}{(x^2 - 48x + 612)^{1/2}} = 0$$

multiply through by the common denominator

Step 3: Find the absolute extrema, checking the endpoints as well as the critical values.

$$400\sqrt{x^2-48x+612} + 250(2x-48)=0$$

$$400\sqrt{x^2-48x+612} + 500x - 12{,}000 = 0$$

$$400\sqrt{x^2-48x+612} = 12{,}000 - 500x$$

$$\sqrt{x^2-48x+612} = 30 - \frac{5}{4}x$$

Step 4: Answer the question posed in the problem.

$$x^2-48x+612 = \left(30-\frac{5}{4}x\right)^2$$

$$x^2-48x+612 = 900 - 75x + \frac{25}{16}x^2$$

$$0 = \frac{9}{16}x^2 - 27x + 288$$

$$0 = 9x^2 - 432x + 4608$$

$$0 = x^2 - 48x + 512$$

$$0 = (x-32)(x-16)$$

reminder
domain
$0 \le x \le 24$

$x=32$, not in domain

$\boxed{x=16}$

From A, move 16 miles toward pt C, then angle underwater toward pt. B.

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Example 6: Linger's Luxury Office Furniture expects to sell 300 executive desks a year. Each desk costs the store \$400, and there is a fixed charge of \$800 per order. If it costs \$200 to store an executive desk for a year, how large should each order be and how often should orders be placed to minimize the inventory costs?

Step 1: Sketch a picture or construct a table to represent the problem and define the variables to represent the unknowns.

x = # desks ordered at 1 time

$\frac{300}{x}$ = # orders placed each year

$\frac{x}{2}$ = avg # desks stored

Step 2: Formulate an equation that will represent the quantity to be maximized or minimized and determine the interval (reasonable domain) over which the function is to be optimized.

Cost of 1 order = $C_O = 400x + 800$

N(number of orders) = $\frac{300}{x}$

$C(\text{of storage}) = C_S = 200 \cdot \frac{x}{2} = 100x$

C (inventory costs) = (# of orders)(\$ of each order) + Storage Costs

$$C_I = \frac{300}{x} \cdot (400x + 800) + 100x$$

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Step 3: Find the absolute extrema, checking the endpoints as well as the critical values.

$$C_I = 120{,}000 + \frac{240{,}000}{x} + 100x$$

$$C_I = 120{,}000 + 240{,}000x^{-1} + 100x$$

$$C_I' = -240{,}000x^{-2} + 100 = 0$$

$$100 = \frac{240{,}000}{x^2}$$

Step 4: Answer the question posed in the problem.

$$100x^2 = 240{,}000$$

$$x^2 = 2400$$

$$x = 48.9897\ldots$$

$$x \approx 50$$

$$N = \frac{300}{50} = 6$$

How large should each order be?

50 desks

How often should orders be placed to minimize costs?

6 orders each year ⇒ every 2 months.

Guidelines for solving word problems:

- Answer the question asked.

  "What is the distance to Aunt Matilda's house?"

  d = distance to Aunt Matilda's house in miles

- Draw a picture, graph or chart, AND

  label the picture with your answer from step one.

- Write an equation.

  Constraint equation: an equation that ties together x and y, for which you can solve for y.

  Objective equation: an equation that represents the quantity that you want to maximize or minimize.

- Solve: to maximize or minimize a quantity, set the derivative equal to zero and solve.

  Check endpoints to see if the max/min value occurs at the critical point or at the endpoint.

- Check

  Make sure you answered the question asked.

  Make sure your answer is in the domain of the original question.

Strategy for Solving Optimization Problems

1. Introduce variables, look for relationships among these variables, and construct a mathematical model of the form

   Maximize (or minimize) $\mathbf{f(x)}$ on the interval $\mathbf{I}$

2. Find the critical values of $\mathbf{f(x)}$.
3. Use the procedures from previous sections to find the absolute maximum (or minimum) value of $\mathbf{f(x)}$ on the interval $\mathbf{I}$ and the value(s) of $\mathbf{x}$ where this occurs.
4. Use the solution to the mathematical model to answer all the questions asked in the problem.