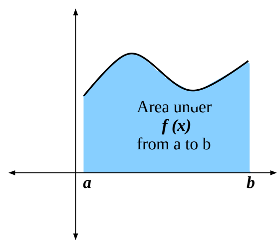
Math142 Lecture Notes
6.4 - The Definite Integral

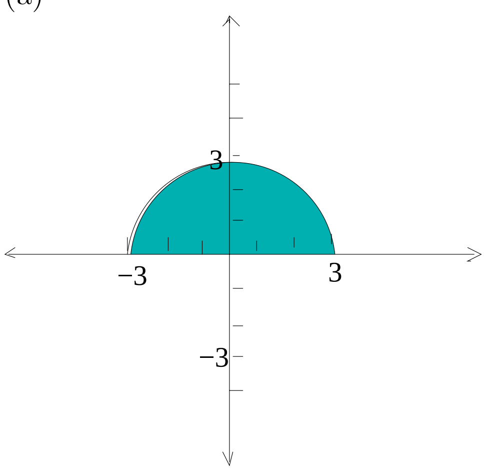
For years, mathematicians tried to find the area under a curve given by the function $y = f(x)$ on the closed interval $[a, b]$.



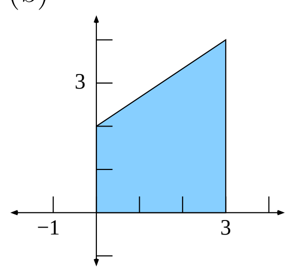
This is not a problem if the area under the curve forms a familiar geometric shape for which a formula for the area has been derived.

Example 1: Find the exact area under the following curves.

(a)



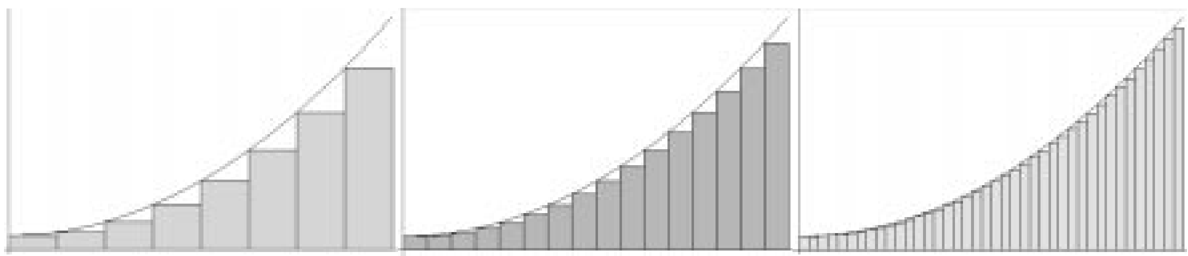
(b)



When the area under a curve is not a familar geometric shape, we use small rectangles to approximate the area under a curve.

Left Sums for $f(x) = x^2 + 1$ on $[0, 4]$

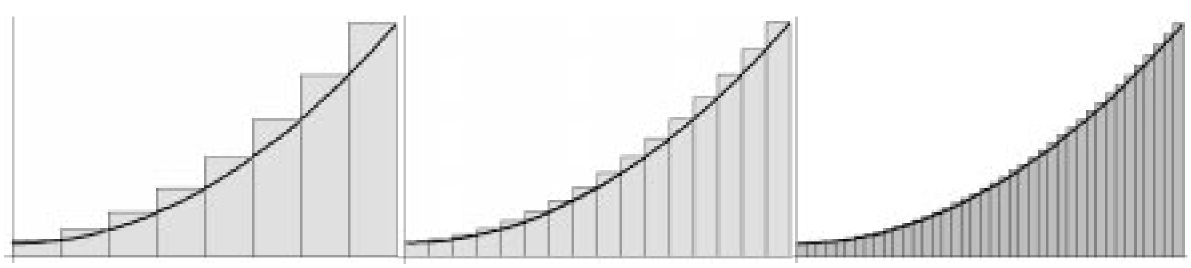
Approximate Area $= \sum_{i=0}^{n-1} \Delta x f(x_i)$ x_i is the left endpoint n is the number of rectangles.



$base = \Delta x =$ $base = \Delta x =$ $base = \Delta x =$

Right Sums for $f(x) = x^2 + 1$ on $[0, 4]$

Approximate Area $= \sum_{i=1}^{n} \Delta x f(x_i)$ x_i is the right endpoint n is the number of rectangles.



$base = \Delta x =$ $base = \Delta x =$ $base = \Delta x =$

Example 2: Calculate the left and right sums for $f(x) = 2x^2$ on $[0, 4]$ using 4 rectangles.

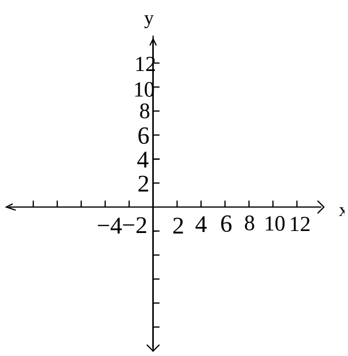
Error Bounds for Approximations of Area by Left or Right Sums
If $f(x) > 0$ and is either increasing, or decreasing on the interval $[a, b]$ then

$$error \leq |f(b) - f(a)| \cdot \frac{b-a}{n}$$

is an error bound for the approximation of the area between the graph of $f(x)$ and the x axis, from $x = a$ to $x = b$.

- Given the function $f(x) = 12 - 0.25x^2$, approximate the area under $y = f(x)$ from $x = 2$ to $x = 6$.

  Graph the function on the interval $[0, 10]$.



  Calculate the left hand sum, L_4, and the right hand sum, R_4, and the error bounds for each.

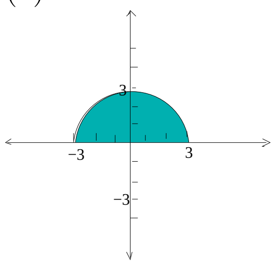
  How large should n be chosen for the approximation of the area by L_4 or R_4 to be within 0.1 of the true value?

Area Under a Curve and the Definite Integral
If f is a continuous function defined on a closed interval $[a, b]$, then the area under the graph of f from a to b is

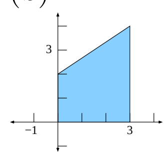
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x = \int_a^b f(x)\ dx$$

$\int_a^b f(x)\ dx$ is called the definite integral with limits of integration a and b.

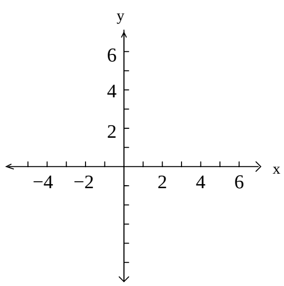
Example 3: Write a definite integral that represents the area shown in example 1.
(a)



(b)

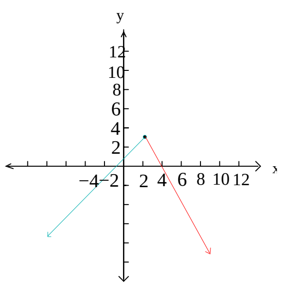


Example 4: Sketch a graph and shade the region given by the definite integral $\int_1^4 \sqrt{x}\ dx$.

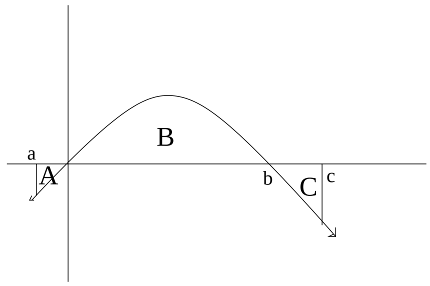


Example 5: Use the rule $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx,\quad a \le c \le b$ and write a definte integral that represents the region of $f(x)$ that lies above the x-axis if

$$f(x) = \begin{cases} x+1 & x \le 2 \\ -2x+7 & x > 2 \end{cases}$$



Example 6: Find the definite integral by referring to the figure below given the area of region A = 1.76, the area of region B = 3.52, and the area of region C = 1.51.



- $\int_a^b f(x)\,dx$
- $\int_0^c f(x)\,dx$

Properties of Definite Integrals

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$, k a constant
4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Example 7. Evaluate each of the following:

a. $\int_2^3 (6x^2)\,dx$

b. $\int_0^2 (9x^2 - 4x)\,dx$

c. $\int_1^4 (12x^3)\,dx$

d. $\int_\star^\triangle (2x^3 + 1)\,dx$