Math142 Lecture Notes
8.2 - Partial Derivatives

How to Compute a Partial Derivative

1. **Partial derivative with respect to** x $\left(f_x(x, y) \text{ or } \dfrac{\partial f}{\partial x}\right)$ - treat y as a constant and then use ordinary derivative techniques. The units of this partial derivative are units of f per units of x.

2. **Partial derivative with respect to** y $\left(f_y(x, y) \text{ or } \dfrac{\partial f}{\partial y}\right)$ - treat x as a constant and then use ordinary derivative techniques. The units of this partial derivative are units of f per units of y.

Example 1: Determine $f_x(x, y)$ and $f_y(x, y)$ if

(a) $f(x, y) = 3x + 2y + 10.$

(b) $f(x, y) = 2x^4 + x^2y^2 - 3y^2 - y$

(c) $f(x, y) = (x^3 - y^2)^4$

(d) $f(x, y) = \dfrac{xy^2}{y - x}$

(e) $f(x, y) = y^2 e^{xy} + \ln x$

Example 2: Determine $f_y(5, 2)$ if $f(x, y) = 37.21x^{0.15}y^{0.87}$. Interpret the meaning of $f_y(5, 2)$.

Example 3: Tube Town, a recently opened water park, spends x thousand dollars on radio advertising and y thousand dollars on television advertising. The park has weekly ticket sales, in tens of thousands of dollars of $TS(x, y) = 1.5x^2 + 3.2y^2$. Determine $TS_x(1, 0.5)$ and $TS_y(1, 0.5)$ and interpret each.

Marginal Productivity of Labor and Capital
For any production of the form $Q = f(x, y) = ax^m y^n$, where $a, m,$ and n are positive constants and x represents units of labor and y represents units of capital, then

- **marginal productivity of labor**, $f_x(x, y)$, gives the approximate change in the productivity per unit change in labor.
- **marginal productivity of capital**, $f_y(x, y)$, gives the approximate change in the productivity per unit change in capital.

Example 4: A golf club manufacturer has a Cobb-Douglas production function given by

$$Q = f(x, y) = 21x^{0.3}y^{0.75}$$

where x is the utilization of labor (in millions), y is the utilization of capital (in millions), and Q is the number of units of golf clubs produced.

(a) Compute $f_x(x, y)$ and $f_y(x, y)$.

(b) If the golf club manufacturer is currently using 150 units of labor and 100 units of capital, determine the marginal productivity of labor and the marginal productivity of capital.

(c) Would production increase more by spending an additional \$1 million on labor or \$500,000 on capital? Explain.

SECOND ORDER PARTIAL DERIVATIVES

If $z = f(x, y)$, then the four possible second-order partial derivatives are

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad f_{yx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 5: Find $f_{xx}(x, y), f_{xy}(x, y), f_{yy}(x, y),$ and $f_{yx}(x, y)$ for $f(x, y) = 3x^4 + 2x^3y^2 - y$.